Explicit mRPI sests for dynamics with zonotopic disturbances

Scientific seminary

Florin Stoican (florin.stoican@acse.pub.ro)

Department of Automatic Control and Systems Engineering, UPB, Romania

Thursday, May 5^{th}, 2016

Outline

Invariance notions

Let us consider the following LTI system in $\mathbb{R}^n$:

$$x^+ = Ax + \delta$$

where δ is a disturbance bounded by $\Delta \subset \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ is a Schur matrix.

Definition

A set $\Omega \subset \mathbb{R}^n$ is called a robust positively invariant set iff $A\Omega \oplus \Delta \subseteq \Omega$.
The mRPI set, $\Omega_\infty(A, \Delta)$, is defined as the RPI set contained in any closed RPI set.

Alternatively, the mRPI set can be seen as the fixed point ($\Omega_\infty(A, \Delta) = A\Omega_\infty(A, \Delta) \oplus \Delta$) associated to the recursion

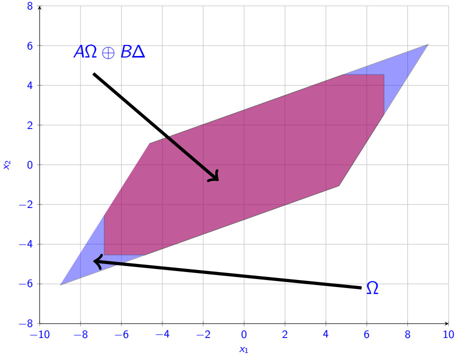
$$\Omega_k(A, \Delta) = A\Omega_{k-1}(A, \Delta) \oplus \Delta, \text{ with } \Omega_0 = \{0\}.$$

and, hence,

$$\Omega_\infty(A, \Delta) \triangleq \lim_{k\to\infty} \Omega_k(A, \Delta) = \bigoplus_{i=0}^{\infty} A^i\Delta.$$

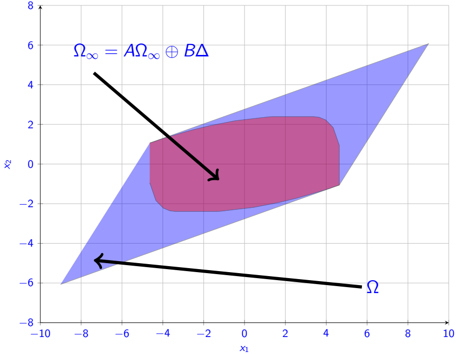
Invariance notions – illustrations

RPI set



$$A\Omega \oplus B\Delta \subseteq \Omega$$

mRPI set



$$\Omega_\infty = \bigoplus_{i=0}^{\infty} A^i B\Delta$$

RPI approximations of the mRPI set

The mRPI set does not have, in general, an analytical representation.

Depending on the set representation and the implementation chosen, we can adopt RPI approximations of the mRPI set:

- iterative approaches
  - inner approximations Raković, Kerrigan, Kouramas, and Mayne 2005
  - outer approximations Olaru, De Doná, Seron, and Stoican 2010
- analytical formulations
  - sublevels of quadratic Lyapunov functions
  - ultimate bounds Kofman, Haimovich, and Seron 2007

Common issues of RPI approximations:

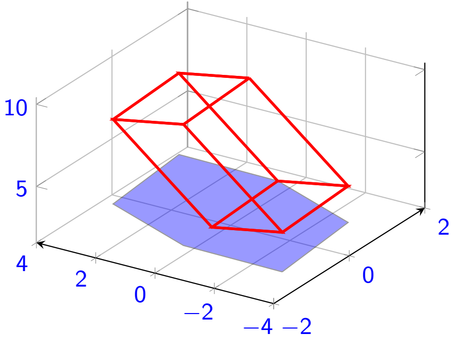
- iterative approaches are precise but are computationally expensive
- Lyapunov functions are conservative

Zonotopic sets

$$Z = \mathcal{Z}\{c, \mathcal{G}\} = \mathcal{Z}\{c, g_1, \ldots, g_m\} = c + \sum_{i=1}^{m} \lambda_i g_i \text{ with } |\lambda_i| \leq 1, \quad \forall i.$$

Representations

- projection of hypercubes
- Minkowski sum of line segments (generators)



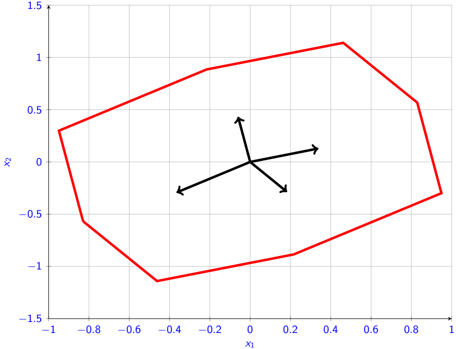
- NP complexity for transformations between representations – Avis and Fukuda 1996
- compact representation in generator form:
  - $2\sum_{i=0}^{n-1}\binom{m-1}{i}$ vertices
  - $2\binom{m}{n-1}$ half-spaces

Zonotopic sets

$$Z = \mathcal{Z}\{c, \mathcal{G}\} = \mathcal{Z}\{c, g_1, \ldots, g_m\} = c + \sum_{i=1}^{m} \lambda_i g_i \text{ with } |\lambda_i| \leq 1, \quad \forall i.$$

Representations

- projection of hypercubes
- Minkowski sum of line segments (generators)



- NP complexity for transformations between representations – Avis and Fukuda 1996
- compact representation in generator form:
  - $2\sum_{i=0}^{n-1}\binom{m-1}{i}$ vertices
  - $2\binom{m}{n-1}$ half-spaces

Zonotopic sets (II)

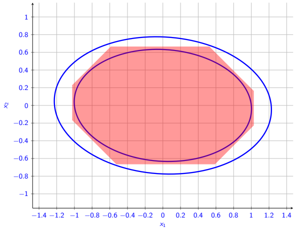
Several properties are of interest for numerical implementations:

- are closed under linear transformation:

$$L\mathcal{Z}\{c, \mathcal{G}\} = \mathcal{Z}\{Lc, L\mathcal{G}\}$$

- are closed under Minkowski sum:

$$\mathcal{Z}\{c_1, \mathcal{G}_1\} \oplus \mathcal{Z}\{c_2, \mathcal{G}_2\} = \mathcal{Z}\{c_1 + c_2, \mathcal{G}_1 \cup \mathcal{G}_2\}$$



- can approximate arbitrarily well any zonoid (Bourgain and Lindenstrauss 1988; Linhart 1989)

Zonotopic sets in control

Zonotopic sets are well-suited to many control problem (usually the noise bounds / constraints are simple and the resulting sets are already zonotopes):

- robust estimation, reachable sets computation – T. Alamo, JM Bravo, and E. Camacho 2005; Le, Stoica, Teodoro Alamo, E. F. Camacho, and Dumur 2013; Althoff and Krogh n.d.; Girard 2005
- MPC strategies – JM Bravo, Limon, T. Alamo, and E. Camacho 2003
- fault diagnosis – Ingimundarson, J.M. Bravo, Puig, T. Alamo, and Guerra 2009; Scott, Findeisen, Braatz, and Raimondo 2014; Blesa, Puig, and Saludes 2011
- many others ...

Exploiting the additional structure given by the zonotopic representation allows to improve upon the explicit descriptions of the mRPI sets Stoican, Hovd, and Olaru 2013; Stoican, Oară, and Hovd 2015.

Outline

mRPI set for zonotopic disturbances

Consider dynamics $x^+ = Ax + \delta$ with zonotopic bounds $\Delta = \mathcal{Z}(c, \mathcal{G})$ and with the associated mRPI set, $\Omega_\infty(A, \Delta)$.

Proposition

The following relations are verified:

i) *for any* $k \geq 1$, $\Omega_\infty(A, \Delta) = \Omega_\infty(A^k, A^{k-1}\Delta \oplus \cdots \oplus \Delta)$;

ii) *given* $\Delta = Z(c, \mathcal{G})$, *the mRPI set can be decomposed as* $\Omega_\infty(A, \Delta) = \left\{(I - A)^{-1} c\right\} \oplus \bigoplus_{i=1}^{m} \Omega_\infty(A, \Delta_i)$, *where* $\Omega_\infty(A, \Delta_i)$ *is the mRPI set associated with dynamics* $x^+ = Ax + \delta_i$; $\delta_i \in \Delta_i \triangleq \{\lambda_i g_i,\ |\lambda_i| \leq 1\}$.

Henceforth we simplify the problem to $x^+ = Ax + g\lambda, \quad |\lambda| \leq 1$, s.t. :

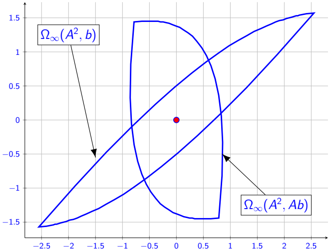
i) the eigenvalues of matrix $A \in \mathbb{R}^{n \times n}$ are real and positive

ii) the disturbance varies along a segment defined by the fixed vector $g \in \mathbb{R}^n$.

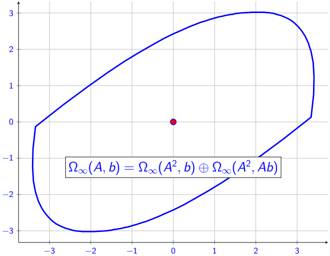
Illustrative example – I

Consider $x^+ = \underbrace{\begin{bmatrix} -1.0559 & 1.1978 \\ -0.1711 & 0.9975 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0.03 \\ 0.31 \end{bmatrix}}_{b} \lambda, \quad |\lambda| \leq 1.$

- for the auxiliary dynamics, '$x^+ = A^2 x + Ab\lambda$' and '$x^+ = A^2 x + b\lambda$'
- obtain the sets $\Omega_\infty(A^2, b)$, $\Omega_\infty(A^2, Ab)$



- combine the resulting mRPI sets to retrieve the set $\Omega_\infty(A, b)$



Extremal trajectories – I

Recall: $\Omega_\infty(A,\Delta) = \lim_{k\to\infty} \Omega_k(A,\Delta)$.

The kth iteration can be seen as

$$\Omega_k(A,\Delta) = \lim_{k\to\infty} \bigcup_{|\lambda_l|\leq 1} \left\{ \sum_{l=1}^{k} A^{k-l} b \lambda_l \right\}$$

The trajectories which reach the boundary (the *extremal trajectories*) are characterized by the extremal sequence $(\lambda_1,\dots,\lambda_k)$ of disturbances:

$$\Sigma = \left\{ \pm(\lambda_1,\lambda_2,\dots) : \lambda_l = \begin{cases} 1, & 1 \leq l < i_1, \\ (-1)^j, & i_j \leq l < i_{j+1}, \\ (-1)^{n-1}, & i_{n-1} \leq l \leq \infty \end{cases} \quad (i_1,\dots,i_{n-1}) \in \mathbf{I}_k^n \right\},$$

where $\mathbf{I}_k^n = \{(i_1,\dots,i_{n-1}) : \ 1 \leq i_1 \leq \dots \leq i_{n-1} \leq k\}$ denotes the collection of all switching sequences of at most '$n-1$' switches happening no later than the k .

In Hu, Lin, and Qiu 2002; Hu, Miller, and Qiu 2002 they were used to characterize null-controllable sets.

Extremal trajectories – II

Lemma

To an extremal sequence $\mathbf{i} = \{i_1, \dots i_{n-1}\}$ *correspond:*

i) *the normal vector* $c_{\mathbf{i}}$ *which respects* $c_{\mathbf{i}}^\top A^{i_j} b = 0, \quad \forall i_j \in \mathbf{i}$;

ii) *extremal point* $x_{\mathbf{i}}^* = \left(2 \sum_{j=1}^{n-1} (-1)^j A^{i_j} + (-1)^n I\right)(I - A)^{-1} b.$

Proposition

The mRPI set is given in

i) *half-space representation (intersection of its extremal half-spaces):*

$$\Omega_\infty(A, \Delta) = \{x : |c_{\mathbf{i}}^\top x| \leq c_{\mathbf{i}}^\top x_{\mathbf{i}}^*, \quad \forall \mathbf{i} \in \mathbf{I}_\infty^n\}$$

ii) *vertex representation (i.e., as convex hull of its extremal points):*

$$\Omega_\infty(A, \Delta) = \underset{\mathbf{i} \in \mathbf{I}_\infty^n}{\mathrm{conv}} \left(\pm x_{\mathbf{i}}^*\right)$$

Extremal trajectories – III

Describing the boundary of the mRPI set through bang-bang sequences allows several remarks:

- three elements interlock to give the boundary description: the switching sequence $\mathbf{i}$, the normal vector $c_{\mathbf{i}}^\top$ and the extremal point $x_{\mathbf{i}}^*$; we may:
  - provide a switching sequence $\mathbf{i}$ and obtain $c_{\mathbf{i}}^\top$ and $x_{\mathbf{i}}^*$;
  - provide a vector $c_{\mathbf{i}}^\top$, obtain the corresponding sequence $\mathbf{i}$ and use it to obtain $x_{\mathbf{i}}^*$.
- the boundary can be covered through bang-bang sequences with $\leq n-2$ switches starting from either of the fix points $x_{e,\pm} = \pm(I-A)^{-1}b$, Hu, Miller, and Qiu 2002;
- hence the boundary *flows* from one fix point to the other De Dona and Lévine 2013
- Still an infinity of points / half-spaces for the mRPI but at least we have an explicit characterization !

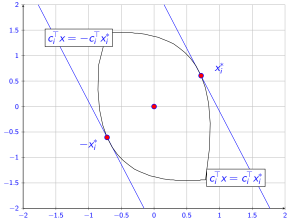
Illustrative example – II (the $\mathbb{R}^2$ case)

The switching sequence becomes $\mathbf{i} = \{i_1\}$, and the sequence of noises becomes

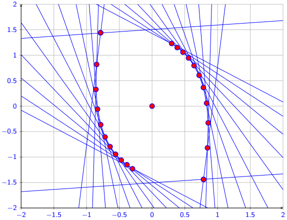
$$\boldsymbol{\Sigma} = \left\{ \pm(\lambda_1, \lambda_2, \dots) : \lambda_l = \left\{ \begin{array}{rl} 1, & 1 \leq l < i_1, \\ -1, & i_1 \leq l \leq \infty \end{array} \right. , i_1 \in \mathbf{I}^2_\infty \right\}$$

- which means that $c_{\mathbf{i}}^\top A^{i_1} b = 0$ and $x_{\mathbf{i}}^* = \pm \left(2A^i - I\right)(I - A)^{-1} b$

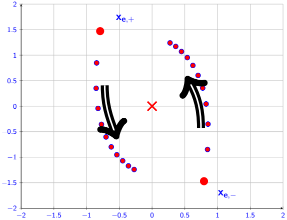
- $x^+ = \begin{bmatrix} 0.91 & -0.07 \\ 0.01 & 0.79 \end{bmatrix} x + \begin{bmatrix} 0.03 \\ 0.31 \end{bmatrix} \lambda$
- switching sequence $\mathbf{i} = \{5\}$ to which corresponds extremal sequence $\{1, 1, 1, 1, -1, -1, \dots\}$
- we obtain $c_{\mathbf{i}} = \begin{bmatrix} 0.7114 \\ 0.2886 \end{bmatrix}$, $\quad x_{\mathbf{i}}^* = \pm \begin{bmatrix} -0.7176 \\ -0.6062 \end{bmatrix}$



Illustrative example – III (the $\mathbb{R}^2$ case)



- half-space representation for switching sequences $\{0, 1, \ldots, 10\} \cup \{\infty\}$



- the boundary is described by two trajectories starting from $x_{e,+}$ and $x_{e,-}$, under constant extremal sequences ($n - 2 = 0$ switches):
  - from $x_{e,-}$ with $\lambda(l) = 1$
  - from $x_{e,+}$ with $\lambda(l) = -1$

Outline

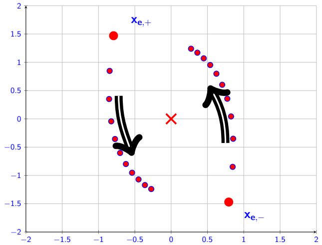
RPI characterizations of the mRPI set – I

We cannot consider all the extremal points (or half-spaces) characterizing the mRPI since there is an infinity of them.

The pairs $\{x_{\mathbf{i}}^*, c_{\mathbf{i}}\}$ agglomerate towards either of the fixed points $x_{e,\pm}$ which means that:

i) we can keep a finite subset $\{x_{\mathbf{i}}^*, c_{\mathbf{i}}\}$ by discarding the ones closer to $x_{e,\pm}$;

ii) this subset is defined by bang-bang sequences happening in a finite time ($\mathbf{i} \in \mathbf{I}_k^n$).



We can obtain RPI approximations of the mRPI set through:

- inner-approximation constructions based on the extremal points
- outer-approximation constructions based on the extremal hyperplanes

RPI characterizations of the mRPI set – II

Consider all extremal switching sequences where the last switch is no later than k:

$$\mathbf{I}_k^n = \{(i_1, \ldots, i_{n-1}) : 1 \leq i_1 \leq \cdots \leq i_{n-1} \leq k\}$$

and the switching sequence

$$\mathbf{I}_e = \{i_1 = \cdots = i_{n-1} = \infty\},$$

which is associated with the fixed points $x_{e,\pm}$.

We may define the following restrictions:

- vertex representation with extremal points:

$$\underline{S}(\mathbf{I}) = \underset{\mathbf{i} \in \mathbf{I}}{\mathrm{conv}} \left(\pm x_{\mathbf{i}}^*\right),$$

- half-space representation with extremal hyperplanes:

$$\overline{S}(\mathbf{I}) = \{x : |c_{\mathbf{i}}^\top x| \leq c_{\mathbf{i}}^\top x_{\mathbf{i}}, \forall \mathbf{i} \in \mathbf{I}\},$$

RPI constructions using inner-approximations

We consider the mRPI restrictions $\underline{S}(\mathbf{I}_k^n)$ and $\underline{S}(\mathbf{I}_k^n \cup \mathbf{I}_e)$.

Proposition

Then there exists the finite scalars $\lambda_k, \mu_k > 1$ *such that:*

i) *the sets*

$$\underline{R}_k(A, \Delta) \triangleq \lambda_k \cdot \underline{S}(\mathbf{I}_k^n)$$

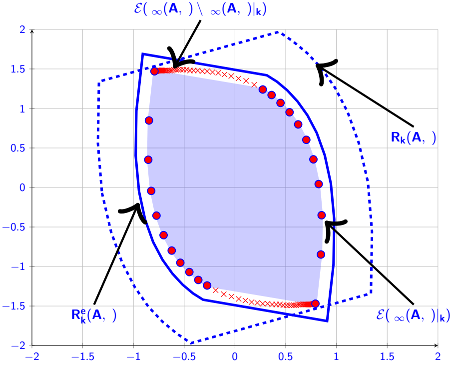
$$\underline{R}_k^e(A, \Delta) \triangleq \mu_k \cdot \underline{S}(\mathbf{I}_k^n \cup \mathbf{I}_e)$$

are RPI;

ii) *the sequences* λ_k, μ_k *are monotonically decreasing for an increasing index* $k > 1$ *and converge towards* 1 *when* k *goes to* ∞.

- since we start with inner-approximations a scaling factor > 1 will always be needed
- $\underline{R}_k^e(A, \Delta)$ converges towards the mRPI much faster than $\underline{R}_k(A, \Delta)$

Illustrative example – IV



Scaling factors λ_k and μ_k for sets $\underline{R}_k(A, \Delta)$ and $\underline{R}^e_k(A, \Delta)$ respectively:

k	5	10	15	20	25	30
λ_k	2.362	1.584	1.454	1.311	1.247	1.247
μ_k	1.830	1.145	1.040	1.012	1.004	1.001

RPI constructions using outer-approximations

We consider the mRPI restrictions $\overline{S}(\mathbf{I}_k^n \cup \mathbf{I}_e)$ and $\overline{S}(\mathbf{I} \cup \mathbf{I}_e)$ with $\mathbf{I} \subset \mathbf{I}_k^n$.

Proposition

For any $k \in \mathbb{N}$,

i) $\overline{R}_k^e(A, \Delta) = \overline{S}(\mathbf{I}_k^n \cup \mathbf{I}_e)$ *is RPI;*

ii) *for a set* $\Omega(\mathbf{I}) = \overline{S}(\mathbf{I} \cup \mathbf{I}_e)$ *there exists* $\gamma \geq 1$ *s.t.* $\gamma\Omega(\mathbf{I})$ *is RPI.*

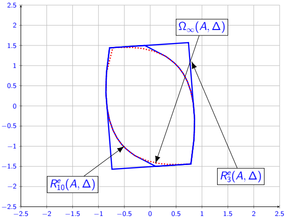
- $\overline{R}_k^e(A, \Delta)$ does not require a scaling factor
- the particular case $\overline{R}_0^e(A, \Delta) = \overline{S}(\mathbf{I}_0^n \cup \mathbf{I}_e) = \overline{S}(\mathbf{I}_e)$ is actually the *ultimate bounds* invariant set Haimovich, Kofman, and Seron 2007;
- this links with Stoican, Olaru, De Doná, and Seron 2011where it was shown that the UB set is tight (it "touches" the mRPI set)

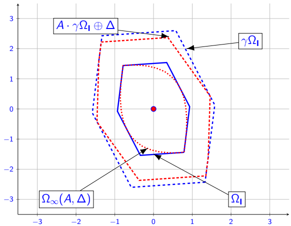
Illustrative example – V

- RPI approximation of the mRPI set using $\overline{R}_3^e(A,\Delta)$ and $\overline{R}_{10}^e(A,\Delta)$
- volume variation for the $\overline{R}_k^e(A,\Delta)$ sets:

k	0	1	2	...	10
$\mathrm{vol}(R_k^e)$	5.62	5.19	4.88	...	4.14
$\Delta\mathrm{vol}(R_k^e)$	*	0.43	0.30	...	0.02
$\frac{\Delta\mathrm{vol}(R_k^e)}{\mathrm{vol}(R_k^e)}$ [%]	*	7.65	5.96	...	0.53

- for $\mathbf{I} = \{5\}$ the scaling factor is $\gamma = 1.6921$





Numerical aspects

- the number of extremal (points) hyperplanes defining the boundary of $\underline{S}(\mathbf{I}_k^n \cup \mathbf{I}_e)$ / $\overline{S}(\mathbf{I}_k^n \cup \mathbf{I}_e)$ is given by:

$$2 \cdot \left(1 + \sum_{i=0}^{n-1} \binom{k}{i}\right)$$

- for a fixed k we have both the inner and outer-approximations which bracket the mRPI set, hence we can estimate the representation's error (via Hausdorff distance, ϵ ball, etc)

- choosing subsets $\mathbf{I} \subseteq \mathbf{I}_k^n$ further reduces the complexity of the representation but requires larger scaling factors

References I

Sasa V. Raković, Eric C. Kerrigan, Koustas I. Kouramas, and David Q. Mayne. "Invariant approximations of the minimal robust positively invariant set". In: *IEEE Transactions on Automatic Control* 50.3 (2005), pp. 406–410.

Sorin Olaru, José A. De Doná, Maŕia M. Seron, and Florin Stoican. "Positive invariant sets for fault tolerant multisensor control schemes". In: *International Journal of Control* 83.12 (2010), pp. 2622–2640. ISSN: 1366-5820.

Ernesto Kofman, Hernan Haimovich, and Maŕia M. Seron. "A systematic method to obtain ultimate bounds for perturbed systems". In: *International Journal of Control* 80.2 (2007), pp. 167–178.

D. Avis and K. Fukuda. "Reverse search for enumeration". In: *Discrete Applied Mathematics* 65.1 (1996), pp. 21–46.

J. Bourgain and J. Lindenstrauss. "Distribution of points on spheres and approximation by zonotopes". In: *Israel Journal of Mathematics* 64.1 (1988), pp. 25–31.

J. Linhart. "Approximation of a ball by zonotopes using uniform distribution on the sphere". In: *Archiv der Mathematik* 53.1 (1989), pp. 82–86.

T. Alamo, JM Bravo, and EF Camacho. "Guaranteed state estimation by zonotopes". In: *Automatica* 41.6 (2005), pp. 1035–1043.

Vu Tuan Hieu Le, Cristina Stoica, Teodoro Alamo, Eduardo F Camacho, and Didier Dumur. "Zonotopic guaranteed state estimation for uncertain systems". In: *Automatica* 49.11 (2013), pp. 3418–3424.

M. Althoff and B.H. Krogh. "Zonotope Bundles for the Efficient Computation of Reachable Sets". In: ().

A. Girard. "Reachability of uncertain linear systems using zonotopes". In: *Hybrid Systems: Computation and Control* (2005), pp. 291–305.

References II

JM Bravo, D. Limon, T. Alamo, and EF Camacho. "Robust MPC of constrained discrete-time nonlinear systems based on zonotopes". In: *European Control Conference*. 2003.

A. Ingimundarson, J.M. Bravo, V. Puig, T. Alamo, and P. Guerra. "Robust fault detection using zonotope-based set-membership consistency test". In: *International Journal of Adaptive Control and Signal Processing* 23.4 (2009), pp. 311–330.

Joseph K Scott, Rolf Findeisen, Richard D Braatz, and Davide M Raimondo. "Input design for guaranteed fault diagnosis using zonotopes". In: *Automatica* 50.6 (2014), pp. 1580–1589.

J. Blesa, V. Puig, and J. Saludes. "Identification for passive robust fault detection using zonotope-based set-membership approaches". In: *International Journal of Adaptive Control and Signal Processing* (2011).

Florin Stoican, Morten Hovd, and Sorin Olaru. "Explicit invariant approximation of the mRPI set for LTI dynamics with zonotopic disturbances". In: *52nd IEEE Conference on Decision and Control*. Florence, Italy, Dec. 2013, pp. 3237–3242.

Florin Stoican, Cristian Oară, and Morten Hovd. "RPI Approximations of the mRPI Set Characterizing Linear Dynamics with Zonotopic Disturbances". In: *Developments in Model-Based Optimization and Control*. Springer, 2015, pp. 361–377.

T. Hu, Z. Lin, and L. Qiu. "An explicit description of null controllable regions of linear systems with saturating actuators". In: *Systems & control letters* 47.1 (2002), pp. 65–78.

T. Hu, D.E. Miller, and L. Qiu. "Null controllable region of LTI discrete-time systems with input saturation". In: *Automatica* 38.11 (2002), pp. 2009–2013.

José A De Dona and Jean Lévine. "On barriers in state and input constrained nonlinear systems". In: *SIAM Journal on Control and Optimization* 51.4 (2013), pp. 3208–3234.

References III

Hernan Haimovich, Ernesto Kofman, and María M. Seron. "Systematic ultimate bound computation for sampled-data systems with quantization". In: *Automatica* 43.6 (2007), pp. 1117–1123.

Florin Stoican, Sorin Olaru, José A. De Doná, and María M. Seron. "Zonotopic ultimate bounds for linear systems with bounded disturbances". In: *Proceedings of the 18th IFAC World Congress*. Milano, Italy, 28 August-2 September 2011, pp. 9224–9229.