On computing the $\mathbb{L}_2$ norm of a generalised discrete–time system

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Abstract—We give explicit analytic formulas for computing the $\mathbb{L}_2$ norm of a discrete–time generalised system whose rational transfer matrix function may be improper or polynomial. The norm is expressed in terms of solutions of generalized Lyapunov equations, written down with coefficients from a special type of realisation of the underlying transfer function matrix, much in the same spirit of the standard (proper) case. The main result hints to a numerically–sound prototype algorithm that relies on standard reliable software for computing solutions of generalised Lyapunov equations.

I. INTRODUCTION

A main motivation for this work comes from some recent advances in the control of generalised (descriptor) discrete–time systems whose transfer matrix functions (tmfs) may be improper or even polynomial, in particular from the optimal $\mathbb{L}_2$ norm control problem. The optimal $\mathbb{L}_2$ norm control problem consists in finding a controller, for a given system, that minimizes the input–output $\mathbb{L}_2$ norm of the resulting system, i.e., minimizes the energy of the output when one applies a Kronecker impulse function to the input. In the standard (proper case), a solution for continuous–time systems was solved in [5] in continuous–time, see for example [4]. The optimal $\mathbb{L}_2$ norm control problem for descriptor systems was solved in [5] in continuous–time and just recently in [10] in discrete–time (see also [2] for several standard control problems formulated for generalised systems).

 Needless to say, when dealing with such problems, an evaluation of the $\mathbb{L}_2$ norm of the underlying systems is required. In [4] an evaluation of the $\mathbb{L}_2$ norm of a proper system is given, both for the stable and anti–stable cases, under the additional assumption that the tmf of the system takes the value zero at infinity (the system is actually strictly proper). Although such an assumption is natural in continuous–time (where, in particular, it is a necessary condition for the finiteness of the norm), in discrete–time it actually restricts the scope of application of the result. The solution proposed in [4] is based on a standard state–space realisation of the system, and solutions of Lyapunov equations having coefficients written in terms of this realisation.

In this paper we take a further step and give an analytic evaluation of the $\mathbb{L}_2$ norm of a completely general discrete–time system, whose tmf is allowed to have poles at infinity, to be even plain polynomial, or to have any zeroes, finite or not. In doing this, we will employ a special type of realisation of the generalised system – called centered – which, apart from modelling the poles at infinity, permits to recover all nice properties of standard state–space realisations.

The paper is organised as follows. In Section 2 we recall several basic properties of centered realisations and two ways in which they could be obtained, either from a descriptor realisation, or directly from the rational tmf expression. Section 3 is devoted to the main results, which are illustrated on two numerical examples in Section 4. The last section draws some conclusions.

II. PRELIMINARIES

By $\mathbb{D}$, $\overline{\mathbb{D}}$, $\mathbb{D}_c$, and $\mathbb{C}_{0;1}$ we denote the open unit disk, its closure, its complement with respect to the one point compactification of the complex plane $\mathbb{C}$, and the unit circle, respectively. By $\Lambda(A – zE)$ we denote the spectrum of $A – zE$ (note that this is a generalised eigenvalue problem, which is, generally, more intricate than solving the standard eigenvalue problem, and can be reduced to the latter one when at least one of $E$ and $A$ matrices are invertible). Given any square matrix $A \in \mathbb{C}^{n \times n}$, define $Tr[A] := \sum_{i=1}^{n} a_{ii}$, where $A := (a_{ij})_{i,j=1;\ldots;n} = A_{ij}$. For the tmf $G(z)$ define its adjoint as $G^*(z) := G^T(\bar{z})$, where the bar takes the complex conjugate of the coefficients.

Further, we recall the notion of centered realisation of a generalised system that have been introduced especially for solving several control problems related to those systems which have poles at infinity, often called improper (for further details, see for example [2], [3] or [7]).

Let $G(z)$ be a general $p \times m$ tmf (possibly improper or polynomial) and let $z_0 := \frac{\alpha}{\beta}$ be any point in $\mathbb{C}$, where $\alpha, \beta \in \mathbb{C}$. Then, $G(z)$ may be represented as

$$G(z) = C(zE – A)^{-1}B(\alpha – \beta z) + D =: \begin{bmatrix} A – zE \quad B \quad C \\ C \quad D \quad z \end{bmatrix} \bigg|_{z_0}$$

which is called a centered realisation (at $z_0$). Here $A, E, B, C, D$ are matrices of dimension $n \times n, n \times n, n \times m, p \times n, p \times m$, respectively, $A – zE$ is a regular pencil, i.e., $\det(A – zE) \neq 0$, and $n$ is called the order (or dimension) of the realisation. The realisation is called minimal if its order is as small as possible. In particular, descriptor realisations (see [6], [2], [5]) are nothing but a particular instance of centered realisations, with $z_0 = \infty$.

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Obviously, we can center the above realisation wherever in the closed complex-plane but, by arguments of symmetry, we will choose to center it on the unit circle. When doing this, we can always choose \( \beta = \alpha \), with \( \alpha \in C_{0(1)} \). Furthermore, a natural choice of \( z_0 \) is outside the set of poles and zeroes of the tfm, and henceforth we assume both these choices are in force throughout the paper.

Centered realisations (with \( z_0 \) outside the set of poles and zeroes of the tfm) have some nice features that make them suitable for the problems at hand, namely:

1) If the realisation is minimal, then its minimal order equals the McMillan-degree of the tfm;
2) \( D = G(z_0) \) (since \( z_0 \) is not a pole) and \( \text{rank}_m(G) = \text{rank}(D) \), where \( \text{rank}_m(\cdot) \) denotes the normal rank of a tfm (since \( z_0 \) is neither a zero);
3) Two minimal realisations of the same tfm are unique up to an equivalence state–space transformation, i.e., if
\[
G(z) = \begin{bmatrix} A_1 - zE_1 & B_1 \\ C_1 & D \end{bmatrix}_{z_0},
\]
are two minimal realisations then there always exist two invertible matrices \( (Q; Z) \) such that \( A_2 - zE_2 = Q(A_1 - zE_1)Z \), \( B_2 = QB_1 \), \( C_2 = C_1Z \);
4) The realisation is called proper if \( (\alpha E - \hat{\alpha}A) \) is invertible. In particular, we can always write a proper realisation (since \( z_0 \) is not a pole), and normalize it by an equivalence transformation such that
\[
(\alpha E - \hat{\alpha}A) = I_n.
\]

In this case, the realisation could be given in terms of solely four matrices, since either \( E \) or \( A \) results from (2).

Centered realisations may be obtained from a descriptor realisation (centered at infinity) and, conversely, any centered realisation may be converted into a descriptor realisation. Indeed, consider \( G(z) \) is given by a descriptor realisation (centered at infinity)
\[
G(z) := \begin{bmatrix} A - zE & B \\ C & D \end{bmatrix},
\]
and fix any \( z_0 \notin \Lambda(A - zE) \). Let \( Q \) and \( Z \) be two invertible (may be chosen orthogonal) matrices such that
\[
Q(A - zE)Z = \begin{bmatrix} A_1 - zE_1 & A_{12} - zE_{12} \\ 0 & A_2 \end{bmatrix},
\]
where \( A_2 \) is an invertible matrix and
\[
\text{rank} \begin{bmatrix} E_1 & E_{12} \end{bmatrix} = \text{rank}E.
\]
Let
\[
\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} := Z^T(A - z_0E)^{-1}B, \quad \begin{bmatrix} C_1 & C_2 \end{bmatrix} := CZ,
\]
where the partitions are conformably to (4). A direct check shows that
\[
G(z) = \begin{bmatrix} A_1 - zE_1 & -E_1B_1 - E_{12}B_2 \\ C_1 & D - C_1B_1 - C_2B_2 \end{bmatrix}_{z_0}
\]
is a proper realisation centered at \( z_0 \). Moreover, provided (3) is minimal, (5) is minimal as well.

Conversely, starting from the following realisation
\[
G(z) := \begin{bmatrix} A - zE & B \\ C & D \end{bmatrix}_{z_0},
\]
assumed to be proper \( (A - z_0E) \) is invertible), it is always possible to construct a descriptor realisation (centered at \( \infty \)). Indeed, since \( A - z_0E \) is invertible we may perform a preliminary state–space equivalence transformation and assume further that the realisation is normalised, i.e.,
\[
G(z) = \begin{bmatrix} I_k + (z_0 - z)E & B \\ C & D \end{bmatrix}_{z_0}.
\]
Let \( r := \text{rank}(E) \). Construct a \( k \times (k - r) \) matrix \( E_{12} \) such that \( \text{rank} \begin{bmatrix} E & E_{12} \end{bmatrix} = k \) and let \( B_1 \) and \( B_2 \) be any solution to the equation
\[
\begin{bmatrix} E & E_{12} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = -B.
\]
Then
\[
G(z) = \begin{bmatrix} I + (z_0 - z)E & (z_0 - z)E_{12} \\ 0 & I \\ C & C_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + \begin{bmatrix} D + C_2B_2 + C_1B_1 \end{bmatrix}
\]
is a descriptor state–space realisation (centered at infinity), where \( C_2 \) is an arbitrary matrix of appropriate dimensions. Moreover, (8) is minimal, provided (7) is.

An alternative method for obtaining centered state–space realisations is two start directly from the rational matrix description of the tfm. To this end, split the \( G(z) \) in two parts
\[
G(z) = G_{sp}(z) + G_p(z),
\]
where \( G_{sp}(z) \) is strictly proper and \( G_p(z) \) is a polynomial matrix. Both \( \text{rmf}s \ G_{sp}(z) \) and \( G_{sp}(z) \) are proper and therefore have minimal standard realisations (centered at infinity)
\[
G_{sp}(z) = \begin{bmatrix} A - \lambda I \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad G_p(1/z) = \begin{bmatrix} N - \lambda I \end{bmatrix} \begin{bmatrix} B_2 \\ C_2 \end{bmatrix},
\]
where \( N \) is a nilpotent matrix. A minimal realisation for \( G(z) \), centered at \( z_0 \) may be written as
\[
G(z) = \begin{bmatrix} A - zI & 0 \\ 0 & I - \lambda N \end{bmatrix} \begin{bmatrix} (z_0I - A)^{-1}B_1 \\ C_2 \end{bmatrix} \begin{bmatrix} B_2 \\ D \end{bmatrix}_{z_0}
\]
where \( D := C_1(z_0I - A)^{-1}B_1 + z_0C_2(I - z_0N)^{-1}B_2 + D_2 \). For more details about these type of conversions see [8], [10].

We conclude this section with the definition of the \( L_2 \) norm

**Definition II.1.** For a general tfm \( G(z) \) (possibly improper and/or polynomial) we define the \( L_2 \) norm by the formula
\[
\|G\|_2^2 := \frac{1}{2\pi} \int_0^{2\pi} \text{Tr}[G^*(e^{i\theta})G(e^{i\theta})]d\theta.
\]
Notice that a necessary and sufficient condition for the existence (finiteness) of the \( L_2 \)-norm of \( G(z) \) is the absence of the \( C_{0(1)} \) poles.
III. MAIN RESULTS

We formulate the main results successively for the stable, anti-stable and dichotomic cases.

Theorem III.1 (The stable case). Let $G(z)$ be a $p \times m$ transfer function given by a normalized centered realization

$$G(z) := \left[ \begin{array}{c} A - zE \\ C \end{array} \right]$$

where $z_0 := \frac{a}{b} \in \mathbb{C}_{0,1}$ is not one of its poles. Assume $A(A - zE) \subset \mathbb{D}$ and, therefore, $E$ is invertible. Then

$$\|G\|^2 = \text{Tr}[B^*Q_EB] + |\alpha|^2\|CE^{-1}B\|^2_{\text{Frobenius}} - 2\text{Re}(\alpha)\text{Tr}[D^*CE^{-1}B] + \|D\|^2_{\text{Frobenius}},$$

where $Q_E$ is the unique positive semidefinite solution of the generalized Lyapunov equation

$$A^*Q_EA - E^*Q_EE + E^{-1}C^*CE^{-1} = 0.$$  \hspace{1cm} (12)

Proof. The dynamics of the system (10) may be written

$$Ex_{i+1} = Ax + Be_i(\alpha \sigma - \bar{\alpha}\sigma_{i+1}), \quad y = Cx + D.$$  \hspace{1cm} (13)

Assume first that $D = 0$. In this case, consider $u_k = \sigma_k e_i$, where $\sigma_k$ is the Kronecker impulse function, and $e_i$ is the $i$-th vector of the canonical base of $\mathbb{R}^n$. Then, from (13) we get

$$Ex_{i+1} = Ax + Be_i(\alpha \sigma - \bar{\alpha}\sigma_{i+1}), \quad y = Cx.$$  \hspace{1cm} (14)

We make the following notations

$$w^i := e_i(\alpha \sigma - \bar{\alpha}\sigma_{i+1}),$$

$$A_E := E^{-1}A,$$

$$B_E := E^{-1}B.$$  \hspace{1cm} (15)

Equation (14) becomes

$$x_{i+1}^i = A_Ex^i + B_Ew^i,$$

$$y^i = Cx^i.$$  \hspace{1cm} (16)

Since $A_E$ is stable (as $A(A - zE) \subset \mathbb{D}$), we have (see (2.148) in [4])

$$x_{i,k}^w = \sum_{j=-\infty}^{k-1} A_E^{k-j-1}B_Ew^i_j,$$

for all $k \in \mathbb{Z}$. Taking into account the special form of $w^i_k$, we get

$$y_{i,k} = Cx_{i,k} = \begin{cases} 0 & \text{if } k < 0, \\ -\bar{\alpha}CBEe_i & \text{if } k = 0, \\ CE^{-1}A_E^{k-1}B_Ee_i & \text{if } k \geq 1. \end{cases}$$

We also have

$$\sum_{i=1}^{m} \|y_{i}^w\|^2 = \sum_{i=1}^{m} \left( \sum_{k=-\infty}^{\infty} e_i^T g_k^w g_k e_i \right) = \sum_{k=-\infty}^{\infty} \left( \sum_{i=1}^{m} e_i^T g_k^w g_k e_i \right)$$

$$= \sum_{k=-\infty}^{\infty} \text{Tr}[g_k^w g_k]$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \text{Tr}[G^*(e^{j\theta})G(e^{j\theta})]d\theta$$

$$= \|G\|^2.$$  \hspace{1cm} (17)

where by $g_k$ we denoted the time-domain input–output operator corresponding to $G(z)$. Further,

$$\|G\|^2 = \sum_{i=1}^{m} \|y_{i}^w\|^2 = \sum_{k=1}^{m} \sum_{i=1}^{\infty} ((y_{i,k}^w)^*)^T y_{i,k}^w$$

$$+ \sum_{k=1}^{m} \sum_{i=1}^{\infty} e_i^T B_E^*(\alpha)^2 C^*C$$

$$+ \sum_{k=1}^{m} \sum_{i=1}^{\infty} (A_E^{k-1}C_E C A_E^{k-1})_E B_E e_i$$

$$= |\alpha|^2 \text{Tr}[B_E^*C^*C B_E] + \text{Tr}[B^*Q_EB],$$  \hspace{1cm} (18)

where, for the last relation see (2.125) in [4]. However, this is exactly our first claim, for the case $D = 0$.

Returning to the general case $D \neq 0$, introduce the notation

$$\Sigma(z) := \left[ \begin{array}{c} A - zE \\ C \end{array} \right]$$

Clearly, $G(z) = \Sigma(z) + D$ and $\Sigma(z)$ is stable as well.

Compute

$$\|G\|^2 = \frac{1}{2\pi} \int_0^{2\pi} \text{Tr}[G^*(e^{j\theta})G(e^{j\theta})]d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \text{Tr}[(\Sigma^*(e^{j\theta}) + D^*)(\Sigma(e^{j\theta}) + D)]d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \text{Tr}[\Sigma^*(e^{j\theta})\Sigma(e^{j\theta})]d\theta$$

$$+ \frac{1}{2\pi} \int_0^{2\pi} \text{Tr}[\Sigma^*(e^{j\theta})D]d\theta$$

$$+ \frac{1}{2\pi} \int_0^{2\pi} \text{Tr}[D^*\Sigma(e^{j\theta})]d\theta + \|D\|^2_{\text{Frobenius}}.$$  \hspace{1cm} (21)

Since $\Sigma(z)$ is stable and has $D = 0$, we may apply formula (19) to compute its norm. It only remains to evaluate the two other terms, i.e.,

$$I_1 := \frac{1}{2\pi} \int_0^{2\pi} \text{Tr}[\Sigma^*(e^{j\theta})D]d\theta,$$

$$I_2 := \frac{1}{2\pi} \int_0^{2\pi} \text{Tr}[D^*\Sigma(e^{j\theta})]d\theta.$$  \hspace{1cm} (22)

For the first integral denote $z := e^{j\theta}$ and compute its differential, namely

$$dz = je^{j\theta}d\theta.$$  \hspace{1cm} (23)
We have
\[ I_1 = \frac{1}{2\pi} \int_0^{2\pi} Tr[\Sigma^*(e^{i\theta})D] d\theta \]
\[ = \frac{1}{2\pi} \int_{C_{0,(1)}} Tr[(\Sigma^*(z)D)z^{-1}] dz \]
However, \( I_1 \) is exactly \( Tr[\Gamma_0] \), where \( \Sigma^*(z)D \) is the \( Z \)-transform of \( \Gamma_k \). Hence, it remains to compute \( \Gamma_0 \). Notice that \( \Sigma^*(z)D \) is an anti-stable system, and writing its dynamic equations leads to
\[ A^*x_{+1} = E^*x + C^*D(\alpha u - \bar{\alpha} u_{+1}), \]
\[ y = B^*x, \]
which are equivalent to
\[ x = E^{-*}A^*x_{+1} - E^{-*}C^*D(\alpha u - \bar{\alpha} u_{+1}), \]
\[ y = B^*x. \]
Denote by
\[ \hat{A}_E^* : = E^{-*}A^* = (AE^{-1})^*, \]
\[ \hat{C}_E^* : = E^{-*}C^* = (CE^{-1})^*. \]
Provided \( u_k \in l^2_n \), it follows from (2.154) in [4] that the inverse–time stable system above has a unique solution, namely
\[ x_{+k}^{u;j} = -\sum_{i=k}^{-\infty} (A_E^{*})^{-1-k} \hat{C}_E^*De_j (\alpha \sigma_i - \bar{\alpha} \sigma_{i+1}), \]
Obviously, \( \dot{x}_{+0}^{u;j} = -\alpha \hat{A}_E e_j \), from where it follows that
\[ \dot{y}_{+0}^{u;j} = -\alpha \hat{C}_E e_j \] and we conclude that
\[ I_1 = -\alpha Tr[\hat{D}^*CE^{-1}B]. \]
For the second integral \( I_2 \) we proceed in the same vein
\[ I_2 = \frac{1}{2\pi} \int_0^{2\pi} Tr[D^*\Sigma^*(e^{i\theta})]d\theta \]
\[ = \frac{1}{2\pi} \int_{C_{0,(1)}} Tr[(D^*\Sigma(z))z^{-1}] dz. \]
Notice that \( I_2 \) is exactly \( Tr[\Gamma_0] \), where \( D^*\Sigma(z) \) is the \( Z \)-transform of \( \Gamma_k \). It remains to compute \( \Gamma_0 \), where now \( D^*\Sigma(z) \) is a stable system. Write down the dynamic equations to get
\[ Ex_{+1} = Ax + B(\alpha u - \bar{\alpha} u_{+1}), \]
\[ y = D^*Cx, \]
which is equivalent to
\[ x_{+1} = E^{-}Ax + E^{-1}B(\alpha u - \bar{\alpha} u_{+1}), \]
\[ y = D^*Cx. \]
Denote by
\[ A_E = E^{-1}A, \]
\[ B_E = E^{-1}B. \]
From (2.148) in [4] it follows that the direct–time stable system above has a unique solution (provided \( u_k \in l^2_n \))
\[ x_{+k}^{u;j} = \sum_{i=-\infty}^{k-1} (A_E)^{k-1-i} B_E e_j (\alpha \sigma_i - \bar{\alpha} \sigma_{i+1}). \]
Clearly, \( x_{+0}^{u;j} = -\bar{\alpha} B_E e_j \), from where we get \( y_{+0}^{u;j} = -\bar{\alpha} D^*C B_E e_j \). Therefore,
\[ I_2 = -\bar{\alpha} Tr[D^*CE^{-1}B], \]
We have thus obtained the following two expressions
\[ I_1 = -\alpha Tr[D^*CE^{-1}B], \]
\[ I_2 = -\bar{\alpha} Tr[D^*CE^{-1}B], \]
which completes the whole proof. \( \square \)

We switch now to the anti–stable case.

**Theorem III.2** (The antistable case). Let \( G(z) \) be a \( p \times m \) tmf given by a normalized centered realisation
\[ G(z) := \begin{bmatrix} A - zE & B \\ C & D \end{bmatrix} z_0, \]
where \( z_0 := \frac{a}{\alpha} \in C_{0,(1)} \) is not one of its poles. Assume \( \Lambda(A - zE) \in D_c \) and therefore \( A \) is invertible. Then
\[ ||G||_2^2 = Tr[C \Sigma_A C^*] + ||A||^2 \frac{CA^{-1}B}{Frobenius} - 2Re(\alpha) Tr[D^*CA^{-1}B] + ||D||_2^2. \]
where \( \Sigma_A \) is the unique positive semidefinite solution of the generalized Lyapunov equation
\[ EP a E^* - AP a A^* + A^{-1}BB^*A^{-*} = 0. \]

**Proof.** Let \( \Lambda(A - zE) \in D_c \). Notice that
\[ ||G||_2^2 := \frac{1}{2\pi} \int_0^{2\pi} Tr[G^*(e^{i\theta})G(e^{i\theta})] d\theta \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} Tr[G(e^{i\theta})G^*(e^{i\theta})] d\theta \]
\[ = ||G||_2^2. \]
Since \( G(z) \) is now assumed to be anti-stable, it follows that \( G^*(z) \) is stable and we apply the above formulas for
\[ G^*(z) = \begin{bmatrix} E^* - zA^* & C^* \\ B^* & D^* \end{bmatrix} z_0, \]
which proves the second assertion of the theorem. \( \square \)

Finally, we state the main result for the general case.

**Theorem III.3** (The dichotomic case). Let \( G(z) \) be a \( p \times m \) tmf without poles on the unit circle. Then there exists a normalized centered pole–separated realisation
\[ G(z) = \begin{bmatrix} A_+ - zE_+ & A_- - zE_- & B_- \\ C_+ & A_+ - zE_+ & B_+ \\ 0 & C_- & D \end{bmatrix} z_0, \]
where \( z_0 := \frac{a}{\alpha} \in C_{0,(1)} \) is not one of its poles, \( (A_- - zE_-) \) is stable, \( (A_+ - zE_+) \) is anti-stable, the generalized Lyapunov equations
\[ A_+^*Q E A_- - E_+^*Q E E_- + E_-^*C_+ C_- E_- = 0, \]
\[ E_+P_+ E_- = A_+ P_+ A_+^* + A_+^-1 B_+ B_+^* A_+^* = 0, \]
have positive semidefinite solutions $Q_E$ and $P_A$, respectively, and the generalized matrix-pencil Sylvester equation

$$U(A_+ - zE_+) + (A_- - zE_-)V + A_\mp - zE_\mp = 0$$

has a unique solution $(U, V)$. Then we have

$$\|G\|_2^2 := \text{Tr}[(B_- + UB_+)Q_E(B_- + UB_+)] + \|\alpha E\|_F^2 \|C_-E_-^{-1}(B_- + UB_+)\|_F^2 - 2\text{Re}(\alpha)\text{Tr}[D^* C_-E_-^{-1}(B_- + UB_+)] + \|D\|_F^2$$

$$+ \text{Tr}[(C_+ + C_-)P_A^*(C_+ + C_-)\ast] + \|\alpha\|_F^2 (C_+ + C_-)A_\mp^{-1}B_\mp \|_F^2.$$

(35)

Proof. This general case follows by combining the stable and anti-stable cases. Indeed, starting from a proper centered realisation

$$G(z) := \begin{bmatrix} A - zE & B \\ C & D \end{bmatrix},$$

where $\Lambda(A - zE) \in \mathbb{C} - \mathbb{C}_{0 \setminus \mathbb{1}}$, we perform a unitary equivalence transformation $(Q, Z)$ such that

$$Q(A - zE)Z := \begin{bmatrix} A_\pm - zE_\pm & A_\mp - zE_\mp \\ 0 & A_\mp - zE_\mp \end{bmatrix},$$

where $\Lambda(A_\pm - zE_\pm) \subset \mathbb{D}$ and $\Lambda(A_\mp - zE_\mp) \subset \mathbb{D}_c$. This transformation can always be achieved (for example by using an ordered $QZ - \text{Algorithm}$). Next, consider the generalized matrix-pencil Sylvester equation (34) which has always a solution $(U, V)$, since the matrix-pencils $(A_\pm - zE_\pm)$ and $(A_\mp - zE_\mp)$ are square, invertible and their spectrums do not intersect (see for example [12]). Performing an additional equivalence transformation (this time non-unitary), with the matrices

$$\begin{bmatrix} I & U \\ 0 & I \end{bmatrix}, \quad \begin{bmatrix} I & V \\ 0 & I \end{bmatrix}$$

on the left- and right-side, respectively, we get

$$G(z) = \begin{bmatrix} A_\pm - zE_\pm & 0 \\ 0 & A_\mp - zE_\mp \end{bmatrix} \begin{bmatrix} B_- + UB_+ \\ C_\mp \\ C_\mp + C_- \mp V \\ D \end{bmatrix}. \tag{36}$$

Denoting

$$G_\pm(z) := \begin{bmatrix} A_\pm - zE_\pm & B_- + UB_+ \\ C_\mp & C_\mp + C_- \mp V \\ 0 & D \end{bmatrix},$$

we get

$$G(z) = G_\pm(z) + G_\mp(z).$$

(37)

where $G_\pm$ is stable and $G_\mp$ is anti-stable. Since $G_\pm$ and $G_\mp$ are orthogonal in the $L_2$ space (see, for example, [11]), we have

$$\|G(z)\|_2^2 = \|G_\pm(z)\|_2^2 + \|G_\mp(z)\|_2^2.$$

Finally, the formula (35) follows by using Theorems III.1 and III.2 on the stable $G_\pm(z)$ and anti-stable $G_\mp(z)$, respectively.

IV. Numerical Examples

In this section we illustrate our results on two numerical examples, one for a simple polynomial (antistable case) and one for a general improper (dichotomic) tfm.

Example 1. Let

$$G(z) := z^2 = \begin{bmatrix} -1 & -z \\ 0 & -1 \end{bmatrix},$$

where we centered the realisation in $z_0 := \frac{2}{5} = \frac{1}{2}$. We underline here that, by arguments of symmetry, we choose $z_0 \in \mathbb{C}_{0 \setminus \mathbb{1}}$ as above. This particular choice of $z_0$ will be insightful in the next developments. Notice that

$$E - A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Therefore, we need further to normalize the realisation by means of an equivalence transformation defined by

$$Q := I_2,$$

$$Z := (E - A)^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$

to get

$$QAZ = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix},$$

$$QEZ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$QB = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$CZ = \begin{bmatrix} 1 & -2 \end{bmatrix}.$$

Compute the unique positive semidefinite solution of the generalized Lyapunov equation

$$E PA^* - AP_A^* + A^{-1} BB^* A^{-\ast} = 0,$$

in the current data

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} PA \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} P_A \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 0,$$

and obtain

$$P_A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}.$$

According to our formula in Theorem III.2 we get

$$\|G\|_2^2 = 1.$$

Example 2. Let $G(z)$ be a $2 \times 2$ tfm given by the realisation

$$G(z) := \begin{bmatrix} \frac{1}{z^2} - 3z + \frac{5}{z^2 - z} + 2z - 10 & \frac{1}{z^2 - z} + \frac{5}{2(z^2 - z)} \\ \frac{1}{z^2} + 3z - \frac{5}{2(z^2 - z)} - 2 - z \\ 0 & 1 - z \\ 0 & 0 \\ 0 & 2 + 2z \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -z & -1 & 0 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$
This realisation is proper and minimal. Therefore, the poles of 
\( G(z) \) are precisely the generalized eigenvalues of the pole pencil 
\( \lambda(A - zE) \), multiplicities counted. The system is indeed

dichotomic since 
\( \lambda(A - zE) = (0; 0; \infty; \infty) \). Consider the

Sylvester equation (34) from Theorem III.3, which in the
current data is

\[
U \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 - z & -1 \\ 0 & 0 & -1 \end{bmatrix} \mathcal{V} + \begin{bmatrix} 1 & z & 0 \\ 0 & z & 0 \end{bmatrix} = 0, \quad (38)
\]
having the unique solution

\[
U = \begin{bmatrix} -1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 1/2 & 0 \end{bmatrix},
\]

\[
V = \begin{bmatrix} -1 & 1/2 & 0 \end{bmatrix}.
\]

Before applying the formula in Theorem III.3 to compute the

\( \mathbb{L}_2 \)-norm of \( G(z) \), we need to normalise the two underlying

subsystems (37). However, \( G_- \) is already normalized since

\( E_- - A_- = 1 - 0 = 1 \), while for \( G_+ \) we apply the

equivalence transformation defined by

\[
Z_+ := I_3,
\]

\[
Q_+ := \begin{bmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix},
\]

to get

\[
\hat{B}_+ := Q_+ B_+ = \begin{bmatrix} -4 & -1 & 1 \\ -3 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix},
\]

\[
\hat{A}_+ := Q_+ A_+ = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix},
\]

\[
\hat{E}_+ := Q_+ E_+ = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Compute the two positive semidefinite solutions of the two

generalized Lyapunov equations (33) and obtain

\[
Q_E := 1
\]

\[
P_A := \begin{bmatrix} 3.4167 & 1.6667 & 2 \\ 1.6667 & 0.9167 & 1 \\ 2 & 1 & 2 \end{bmatrix}.
\]

Now, apply relation (35) to obtain 
\( \| G \|_2^2 = 113.8333 \), from

where we finally get

\[
\| G \|_2 = 10.6693.
\]

V. Conclusion

We managed to give numerically–sound analytic formulas

for computing the \( \mathbb{L}_2 \)-norm of a completely general discrete–
time system given by a centered realisation. The formulas

rely on solutions of generalized Lyapunov equations and essen-
tially regain the same simplicity of the standard (proper) case.