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\( \mathcal{H}_2 \) optimal control for generalized discrete-time systems

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ABSTRACT

For a generalized discrete-time system, whose transfer function matrix may be improper or polynomial, we give realization based formulas for the \( \mathcal{H}_2 \) optimal output feedback controller. The formulas bear essentially the same elegant simplicity of the proper case.

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1. Introduction

The celebrated \( \mathcal{H}_2 \) optimal control has received much attention in the past few decades, motivated mainly by its keystone role played in a wide range of topics in modern control theory (for a comprehensive overview see Saberi, Sannuti, and Chen (1995)). The focus of this work is on \( \mathcal{H}_2 \)-optimal output feedback control wherein the design problem consists in minimizing the \( \mathcal{H}_2 \) norm of a certain transfer function matrix (TFM) while utilizing a measurement feedback controller which guarantees closed-loop stability. Various mathematical techniques have been employed so far for solving this design problem for different classes of systems, either in a stochastic or deterministic context: the model matching technique applied for a multivariable system, either in state-space or TFM settings (Anderson & Moore, 1989; Doyle, 1984; Francis, 1982; Vidyasagar, 1985), the Diophantine equation for a scalar system (Kucera, 1986), the generalized Popov based approach for continuous- and discrete-time systems (Ionescu, Oară, & Weiss, 1999), to name just a few.

More recently, the model matching technique has been extended in Takaba and Katayama (1998) to cover the case of a generalized continuous-time system whose TFM may be improper. Improper systems provide a great tool for modeling general physical systems, since they can include algebraic non-dynamic constraints, impulse behavior and reversed time dynamics (Dai, 1989). The range of applications of these systems varies from engineering, including power systems, electrical networks, aerospace industry, control of robots, modeling and control of algebraic dynamical systems, mechanical systems (see for example Gunther and Feldmann (1999); Lind and Schmidt (2002); Rabier and Rheinboldt (2000)), to economics (Luenberger, 1977).

Motivated by this wide applicability, we extend here the optimal \( \mathcal{H}_2 \) control theory to cover the class of generalized discrete-time systems and provide an explicit construction of the optimal controller. Our approach based on the generalized Popov theory (Ionescu et al., 1999) leads to a realization based solution expressed in terms of two special algebraic Riccati equations.

The paper is organized as follows. Section 2 gives the general framework. In Section 3 we present the main result and defer its proof to the Appendix. We work out an illustrative numerical example in Section 4. The paper ends with several conclusions.

2. Preliminaries

By \( \mathbb{C} \), \( \mathbb{D} \), and \( \mathbb{D}^\infty \) we denote the complex plane, the open unit disk, and the unit circle, respectively, and let \( \mathbb{C}^* := \mathbb{C} \cup \{\infty\} \) be the one point compactification of the complex plane. For a constant matrix \( A \) with elements in \( \mathbb{C} \) we denote by \( A^* \) its conjugate transpose. If \( A \) is invertible, \( A^{-*} \) is its conjugate transpose inverse. We will use \( \Lambda(A - z E) \) to denote the union of generalized eigenvalues of the matrix pencil \( A - z E \) (finite and infinite, multiplicities...
counting). The set of \( p \times m \) TFMs with rational elements having coefficients in \( \mathbb{C} \) is denoted by \( \mathbb{C}^{p \times m}(z) \).

To represent an improper \( T \in \mathbb{C}^{p \times m}(z) \) and to make formulas and numerical computations amenable, we will use a general type of realization called centered:

\[
T(z) = D + C(zE - A)^{-1}B(zI - zE = A) =: \begin{bmatrix} A - zE & B \\ C \\ D \\ z \end{bmatrix},
\]

where \( z_0 \in \mathbb{C} \) is fixed, \( z \) is called the order (or the dimension) of the realization, \( A, E \in \mathbb{C}^{n \times n}, B \in \mathbb{C}^{n \times m}, C \in \mathbb{C}^{p \times n}, D \in \mathbb{C}^{p \times m} \), and the matrix pencil \( A - zE \) is regular, i.e., \( \det(A - zE) \neq 0 \). A realization (1) is called minimal if its order is as small as possible among all realizations of this type. Centered realizations have been extensively used in the literature to solve problems for generalized systems whose TFMs are improper (Gohberg & Kaashoek, 1988; Gohberg, Kaashoek, & Ran, 1992; Oară & Sabău, 2011; Rakowski, 1992). Centered realizations have several advantages over generalized (descriptor) realizations (Verghese, Van Dooren, & Kailath, 1979) (which may be seen as realizations centered at \( z_0 = \infty \)) due to the flexibility in choosing \( z_0 \) always disjoint from the set of poles of \( T \), e.g., the order of a centered minimal realization always equals the McMillan degree of \( T \) and the matrix \( D \) in (1) equals the value of \( T \) in \( z_0 \). Throughout this paper, we will consider realizations centered on the unit circle, i.e., \( z_0 = \mathbb{D} \), with \( z_0 \) not a pole of \( T \). Moreover, to improve the numerical-roundness of the subsequent computations, a convenient \( z_0 \) should be chosen well distanced from the poles of \( T \), which ensures a well-conditioned matrix \( zE - A \) with respect to inversion.

Switching back and forth between generalized (descriptor) realizations and centered realizations can be done by some simple formulas (see Section 5 in Oară and Sabău (2009)). For example, for \( T \in \mathbb{C}^{p \times m}(z) \), we show now a direct method to obtain a minimal realization centered at \( z_0 \in \mathbb{C} \) (which is not among its poles). Indeed, we can always write \( T(z) = G(z) + P(z) \), where \( G(\infty) = 0 \) and \( P(z) \) is a matrix polynomial. Minimal realizations for \( G(z) \) and \( P(z) \) may be obtained with numerically reliable algorithms in the form (see Dai (1989)):

\[
G(z) = \begin{bmatrix} A - zL \\ C_1 \\ 0 \\ I - zN \\ C_2 \\ z \end{bmatrix}, \quad P(1/z) = \begin{bmatrix} N - zI \\ B_2 \\ D_2 \\ z \end{bmatrix},
\]

where \( N \) is a nilpotent matrix. Then it can be directly checked that

\[
T(z) = \begin{bmatrix} A - zL \\ C_1 \\ 0 \\ I - zN \\ C_2 \\ z \end{bmatrix} \begin{bmatrix} z(I - A)^{-1}B_1 \\ 0 \\ (I - \alpha N)^{-1} \end{bmatrix} =: \begin{bmatrix} A - zE \\ B \\ C \\ D \\ z \end{bmatrix},
\]

is a minimal realization centered at \( z_0 \), with \( D := \begin{bmatrix} C_1 (I - A)^{-1}B_1 + z_0 C_2 (I - \alpha N)^{-1}B_2 + D_2 \end{bmatrix} \).

We call the system (1) stable if its pole pencil \( A - zE \) has \( \Lambda(A - zE) \subset \mathbb{D} \), see e.g., Dai (1989). A realization (or the pair \( (A, zE, B) \)) is called stabilizable if: (i) rank \( A - zE \) = \( n, \forall z \in \mathbb{C} \setminus \mathbb{D} \); (ii) rank \((E \quad B) = n \). Analogously, we say that a realization (1) is detectable (or the pair \( C, A - zE \) is detectable) if \((A^* - zE^*, C^*) \) is stabilizable.

Finally, define the \( J_2 \)-norm of a stable system (1) as

\[
\|T\|_2^2 := \frac{1}{2\pi} \int_0^{2\pi} \text{Trace}\left[T^*(e^{i\theta})T(e^{i\theta})\right] d\theta.
\]

A useful evaluation of the \( J_2 \)-norm of the system \( T(z) \), with input \( u \) and output \( y \), is given by

\[
\|T\|_2^2 = \sum_{i=1}^{m} \|y^i\|_2^2,
\]

where \( y^i, i = 1, \ldots, m \) is the componentwise output for the componentwise input \( u^i := d_i, i = 1, \ldots, m \), with \( d \) the discrete-time unit impulse and \( e_{1:n}^T \) the Euclidean basis of \( \mathbb{R}^n \).

### 3. Main result

Consider a general system \( T \in \mathbb{C}^{p \times m}(z) \) (possibly improper or polynomial), having input \( u \) and output \( y \), written in partitioned form

\[
\begin{bmatrix} y_1 \\ y_2 \\ \vdots \end{bmatrix} = T \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix},
\]

where \( T_{ij} \in \mathbb{C}^{p \times m}(z) \) with \( i, j \in \{1, 2, m\}, m := m_1 + m_2, p := p_1 + p_2 \). The \( J_2 \)-optimal control problem consists in finding an appropriate controller \( K \in \mathbb{C}^{m \times p}(z), u_2 = Ky_2 \), such that the closed-loop system, i.e., \( T_{12} = \text{LFT}(T, K) := T_{11} + T_{12}K(I - T_{22}K)^{-1} \), is internally stable and has minimum \( J_2 \) norm, i.e., \( \|T_{12}\|_2 \) attains its minimum over the class of internally stabilizing controllers. Here follows the main result.

**Theorem 1.** Let

\[
T(z) = \begin{bmatrix} A - zE \\ B_1 \\ B_2 \\ C_1 \\ C_2 \\ D_1 \\ D_2 \\ z \end{bmatrix},
\]

Assume the following holds for \( H_1 \):

\( (A - zE, B_2) \) is stabilizable and \( \forall \theta \in [0, 2\pi) \),

\[
\text{rank}(A - \theta^\ast B_2) = n + m_2.
\]

Then the descriptor discrete-time algebraic Riccati equations (DDTAREs) (see Oară and Andrei (2013))

\[
E^\ast XE - A^\ast XA - ((zE - A)X + L^\ast C_2)D_1 \subset \mathbb{D} \quad \text{and} \quad \text{rank}(E \quad B) = n.
\]

are the stabilizing Riccati feedbacks. Furthermore, there exists a controller

\[
K(z) = \begin{bmatrix} A - zE + (B_2F + L^\ast C_2)(zE - A) \\ -L^\ast \end{bmatrix}
\]

that solves the \( J_2 \)-optimal control problem.

**Remark 2.** The hypotheses \((A - zE, B_2) \) stabilizable and \((C_2, A - zE) \) detectable are necessary conditions for the existence of a stabilizing controller. Indeed, it is well-known in the proper case that \( K \) stabilizes \( T \) iff it stabilizes \( T_{12} \). The proof in the general case follows mutatis mutandis from Chapter 6, Zhou, Doyle, and Glover (1996), see also Oară and Sabău (2011).

**Remark 3.** \((H_1) \) and \((H_2) \) are regularity assumptions, see Saberi et al. (1995); Zhou et al. (1996) for the standard case. The conditions (7) and (8) ensure that \( T_{12} \) and \( T_{11} \) have no zeros on the unit circle. If either of these two assumptions does not hold, we get a singular \( J_2 \)-optimal control problem in which the optimum may even not be reached (see for example Stoovogel (1992) for the proper...
case) and which is beyond the scope of this paper. In particular, (H1) shows that
\[
\begin{bmatrix}
A - e^{\theta_0 E} & 0 \\
C_1 & D_{12}
\end{bmatrix} = n + m_2 \Rightarrow \text{rank } D_{12} = m_2,
\]
where \(\theta_0\) is such that \(z = e^{\theta_0 t}\). Thus \(D_{12}^* D_{12}\) is invertible. Further, this implies that the DTDARE (9) is well defined and has a unique stabilizing Hermitian solution, see the Appendix in Oară and Andrei (2013). Moreover, numerically sound formulas for the stabilizing solution to the DTDARE (9) may be obtained from Theorems 10, 11 in Oară and Andrei (2013). Dual conclusions follow from (H2).

Remark 4. In Theorem 1 we have implicitly assumed that \(T_{12}(z_0) = D_{11} = 0\) and \(T_{22}(z_0) = D_{32} = 0\). The assumptions are made for simplifying the formulas, with no loss of generality. Indeed, if \(K\) is a solution to the problem with \(D_{22} = 0\), then \(K(f + D_{22} K)^{-1}\) is a solution for the original problem. Further, assume \(D_{11} \neq 0\) in which case
\[
T(z) = \begin{bmatrix}
A - zE & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & 0
\end{bmatrix}.
\]
(13)

Since (H1) and (H2) hold, define the constant feedback matrix \(R = -(D_{11} D_{12})^{-1}(D_{11} D_{12}^*) (D_{21} D_{22}^*)^{-1} \in \mathbb{C}^{p \times m_2}\) to obtain a new system \(\hat{T}\) having the inputs \(y_1\) and \(\epsilon = y_2 - R y_2\), the same outputs as the original system, and \(D_{12} := D_{11} + D_{12} R D_{21} = 0\). Write down the controller \(K\) in (12) for \(T\). Connect now \(K\) with \(-R\) to obtain the controller in (14). It can be easily checked that the closed-loop system with (13) and \(K\) in Eq. (14) is stable. Moreover, since the constant feedbacks cancel each other, the controller in Eq. (14) solves the \(\mathcal{H}_2\) optimal control problem for (13). A similar technique is employed in the standard case (Chapter 14.7, Zhou et al. (1996)).

4. A numerical example

We exemplify our results on a system having both polynomial and improper elements. Let
\[
T(z) = \begin{bmatrix}
-2z^2 + 4z - 1 & -z^2 + 2z - 4 + 4z & z^2 - 2z^2 + 3z + 2 \\
1 & -z^2 + z - 2 & z^2 - 3z + 2 \\
0 & -z^2 + 4z + 1 & 2z^2 - 4z + 3z + 1
\end{bmatrix},
\]
with \(m_1 = p_1 = 2\). A realization centered in 1 is
\[
T(z) = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & z & 1 & z & 1 & 0 \\
-3 + z & 0 & 0 & 2 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & -2 & 2 \\
0 & 0 & 1 & 0 & 1 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -2 & 1 & -1 & 1 & 1 & 0
\end{bmatrix}.
\]
The system has 3 poles at \(\infty\) (one with multiplicity 2 and one with multiplicity 1) and one simple pole on the unit circle in \(-1\). Since \(D_{11} = T_{11}(1) \neq 0\), we use the expression of the controller in Eq. (14). The computed stabilizing solutions of the DTDAREs (9) and (10) are given in Eq. (15), together with the corresponding stabilizing Riccati feedbacks \(F\) and \(L\).
\[
x = \begin{bmatrix}
-29.5181 & -11.6722 & -57.0362 & 42.8142 \\
-11.6722 & -6.7200 & -24.3445 & 17.4688 \\
-57.0362 & -24.3445 & -114.0724 & 87.6284 \\
42.8142 & 17.4688 & 87.6284 & -73.6786
\end{bmatrix},
\]
\[
y = \begin{bmatrix}
-1.3576 & -0.5375 & -2.1399 & 0.1652 \\
-0.5375 & -62.3754 & -114.4883 & 20.0057 \\
-2.1399 & -114.4883 & -7.6695 & 1.7845 \\
0.1652 & 20.0057 & 1.7845 & -7.9661
\end{bmatrix},
\]
(15)
\[
F = \begin{bmatrix}
\end{bmatrix},
\]
\[
L = \begin{bmatrix}
0.7071 & -4.3664 & -0.9229 & 4.7806
\end{bmatrix},
\]
(16) \(R = -0.60\), the controller is
\[
K(z) = \frac{-0.0863 z^4 - 0.1409 z^3 - 0.2232 z^2 + 0.1449 z + 0.2893}{z^4 - 1.345 z^3 - 0.3514 z^2 + 1.012 z - 0.2893},
\]
and the closed-loop system given in Eq. (16) in Box I is proper and stable, having all poles \((0.6893, 0.2853 \pm 0.3367 i, 0.3792, 0, -0.3820, -0.8894)\) inside the unit circle. The optimal value of the \(\mathcal{H}_2\)-norm is \(\|T_{12}\|_2 = 79.1190\).

5. Conclusions

The \(\mathcal{H}_2\) optimal control problem for generalized discrete-time systems (possibly improper or polynomial) has been solved by avoiding the explicit reduction to a model-matching problem and inner–outer factorizations. A realization based formula for the optimal controller which resembles the standard case was given in terms of stabilizing solutions to two special Riccati equations, leading to a numerical burden of the overall algorithm similar to the proper case.

Appendix

The proof is based on the solution of two particular problems, each stated as a separate proposition. The problems will be called one-block and two-block, borrowing the terminology from the model matching problem.

Consider first the one-block problem, for which \(p_1 = m_2, p_2 = m_1 (D_{12} \text{ and } D_{21} \text{ are square}, and)
\]
\[
(A_1) \ D_{12} \in \mathbb{C}^{m_2 \times m_2}\text{ is invertible and } \Lambda (A - zE - B_2 D_{12} D_{12}^*) C_1 (z_0 - z) \subset \mathbb{D}.
\]
\[
(A_2) \ D_{21} \in \mathbb{C}^{m_1 \times m_1}\text{ is invertible and } \Lambda (A - zE - B_1 D_{12} D_{12}^*) C_2 (z_0 - z) \subset \mathbb{D}.
\]

Note that \((A_1)(A_2)\) is equivalent to \(T_{12}(z_0)\) invertible, having only stable zeros.

Proposition 5. For the one block problem \(\min_K \|T_{12}\|_2 = 0\) and the \(\mathcal{H}_2\) optimal controller is given by
\[
K(z) = \begin{bmatrix}
A - zE - (B_2 D_{12} D_{12}^*) C_1 (z_0 - z) & B_2 D_{12} D_{12}^* \\
-D_{12} D_{12}^* C_1 & 0
\end{bmatrix},
\]
(18)
\[
K(z) = \begin{bmatrix}
A - zE + (B_2 F + L^* C_2) (z_0 - z) & L^* \\
F & 0
\end{bmatrix},
\]
(19)

where \(F\) and \(L\) are stabilizing feedbacks, i.e., such that \(\Lambda (A - zE + B_2 F (z_0 - z)) \cup \Lambda (A - zE + L^* C_2 (z_0 - z)) \subset \mathbb{D}\). For the one-block
problem, take in particular $F := -D_{12}^{-1}C_1$, $L^* := -B_1D_{21}^{-1}$ (which are obtained from (11) for $X = Y = 0$). According to (A1) and (A2), $F$ and $L$ are stabilizing feedbacks, leading to the internally stable system in (Eq. 17) in Box I. Actually, substituting these $F$ and $L$, we get precisely the controller in (18). Further, insert $F$ and $L$ in (Eq. 17) in Box I and remove the stable uncontrollable and unobservable parts to get $T_{CL} = 0$ which shows that the minimum $H_2$ norm is attained in zero.

$$\|y_{2,1}\|^2 = \|(C_k x + D_{12} u_{2,k})^*(C_k x + D_{12} u_{2,k})\|$$

$$= \|z_{2,1}\|^2 + 2\Re (\bar{x}^* x u_{2,k}^* x u_{2,k})$$

$$= \|z_{2,1}\|^2 - (E_k + B_2 u_{2,k})^* \Re (\bar{x}^* x u_{2,k}^* x u_{2,k}) + (A_{X_k} + z_0 B_{02} u_{2,k})^* \Re (\bar{x}^* x u_{2,k}^* x u_{2,k}).$$

Consider now the two-block problem, for which $p_2 = m_1, T_{21}$ fulfills the hypothesis (H.1) in Theorem 1, and $T_{21}$ fulfills hypothesis (A.2) from the one block problem.

**Proposition 6.** For the two block problem the DDTARE (9) has a stabilizing Hermitian solution $X$, the $H_2$ optimal controller is given by

$$K(z) = \left[ \begin{array}{c} A - zE + (B_2 F - B_1D_{21}^{-1}C_2)(z_0 - z) & z_{03}^{-1} \end{array} \right],$$

$$\min_k \|T_{CL}\|_2^2 = 2 \text{ Trace } [\Re (B_2^* C_1^* E_1 F^*) - B_2^* X B_1^*].$$

where $F$ is the corresponding stabilizing Riccati feedback in (11), and $A_F = A + z_0 B_2 F, E_2 = E + B_2 F$.

**Proof.** From (H.1) it follows that the DDTARE (9) has a stabilizing Hermitian solution $X$, see Appendix in Oarănd Andrei (2013). It can be easily seen that the existence of $X = X^*$ for the DDTARE (9) is equivalent to the existence of a solution $(X = X^*, V, W)$ to the system of equations

$$\begin{align*}
D_{12}^* X_{12} &= V^* W, \\
(z_0 E - A)^* X_{12} B_2^* + C_1^* D_{12}^* &= W^* V, \\
E^* X_{12}^* A^* X_{12}^* + C_1^* C_1 &= W^* W,
\end{align*}$$

with $V \in \mathbb{C}^{m_1 \times m_2}$, $V \in \mathbb{C}^{m_2 \times m_1}$, and $F := -V^{-1} W$ the corresponding stabilizing feedback. Replace the output $y_1 = C_k x + D_{12} u_{2} \in (6)$ with $\tilde{y}_1 = W x + V u_{21}$ to obtain a new system

$$\tilde{T}(z) = \left[ \begin{array}{c} \tilde{T}_{11} & \tilde{T}_{12} \\
\tilde{T}_{21} & \tilde{T}_{22} \end{array} \right] = \left[ \begin{array}{c} A - zE & B_1 \\
0 & V \end{array} \right],$$

Since $V$ is invertible and the pencils $A - zE - B_1 D_{21}^{-1} C_2 (z_0 - z)$, $A - zE - B_2 V^{-1} W (z_0 - z)$ are stable, $T$ satisfies (A2) and (A1) of the one block problem. Write the controller (18) for $T$ in (25) to obtain (22). In other words, the controller (22) solves the one block problem for $T$. The internal stability of the closed-loop system $T_{CL} = LFT(T, K)$ follows from the internal stability of $T_{CL} = LFT(T, K)$, since the two systems share the same pole pencil.

It remains to prove the optimality requirement. We evaluate successively in (20), the $H_2$-norm. Let $u_{2,k} = u_{1,k}^* := \delta_i e_i$ for $i = 1, \ldots, m_1, k \geq 0$, where $e_i$ is the canonical base of $\mathbb{R}^{m_1}$ and $\delta_i$ the discrete-time unit impulse. From the dynamics of the general system (6),

$$E_{02} + B_2 u_{2,k} = A x_{k-1} + B_1 e_i + A_k z_0 B_{02} u_{2,k-1},$$

with $A_k := z_0 B_{02} - \delta_i$. Substitute (26) in (20) and sum the resulting expression from $k = 0$ to $\infty$ to obtain the first equality in (21). Note that $\lim_{m_1 \to \infty} x_0 = 0$, since the closed-loop system is stable. From the dynamics of the controller (22) we can write $u_{2,k} = F_k e_i$, where $e_i$ is the state of the controller. Since the hypothesis (A1) is fulfilled, $\| \mathbf{X} \|_F = \infty$. Thus $u_{2,0} = F_0 e_i$ and we further get the last equality in (21). Here, $x_0 = -E_{01}^{-1} B_1 e_i$ is the initial condition, derived from the backward dynamics. Introduce $x_0 \in (21)$, sum both sides from $i = 1, \ldots, m$, and recall the evaluation (4) to arrive at $\|T_{CL}\|_2^2 = \|T_{CL}\|_2^2 + 2 \text{ Trace } [\Re (B_2^* C_1^* E_1 F^*) - B_2^* X B_1^*]$. The equality holds for all internally stabilizing controllers. Combining this with Proposition 5 which shows that $\min_k \|T_{CL}\|_2^2 = 0$ is attained for $K$ in (22), we finally get for $K$ the evaluation (23).

**The next result follows by duality from Proposition 6.**

**Proposition 7.** Assume $p_1 = m_2, (A_1)$ and (H.2). Then the DDTARE (10) has a stabilizing Hermitian solution $Y$, the $H_2$ optimal controller is given by $K(z)$

$$K(z) = \left[ \begin{array}{c} A - z E + (B_1^* F - B_2^* D_{12}^{-1} C_2)(z_0 - z) & z_{03}^{-1} \end{array} \right],$$

$$\min_k \|T_{CL}\|_2^2 = 2 \text{ Trace } [\Re (C_1 E_1^{-1} A Y C_1^* - C_1 Y C_1^*)],$$

where $L$ is the corresponding Riccati stabilizing feedback in (11), $A_L := A + z_0 L^* C_2$ and $E_L := E + L^* C_2$.

We finally proceed with:

**Proof of Theorem 1.** The idea of the proof is to reduce the problem to a dual two-block one. We will follow closely the main guidelines of the proof of Proposition 6. With (H.1) and (H.2) it follows that the DDTAREs (9) and (10) have stabilizing Hermitian solutions. The DDTARE (9) has a stabilizing solution iff the matrix system (24) has a stabilizing solution $(X = X^*, V, W)$, with $F = -V^{-1} W$. Replace the output $y_1 = C_k x + D_{12} u_{2} \in (6)$ with $\tilde{y}_1 = W x + V u_{11}$ to obtain the system $T$ in (25). Since $V$ is invertible and $A (A - z E - B_2 V^{-1} W (z_0 - z)) \in C, T$ satisfies (A1) and (H.1). Therefore, we have

$$T(z) = \left[ \begin{array}{c} T_{11} & T_{12} \\
T_{21} & T_{22} \end{array} \right] = \left[ \begin{array}{c} A - zE & B_1 \\
0 & V \end{array} \right],$$

$$\|T_{CL}\|_2^2 = \|T_{CL}\|_2^2 = 2 \text{ Trace } [\Re (C_1 E_1^{-1} A Y C_1^* - C_1 Y C_1^*)].$$
a dual two-block problem for $\tilde{T}$ solved in Proposition 7. Write the controller (27) for the system $T$ to obtain (12). By a similar argument, the closed-loop system $T_{CL} = \text{LFT}(T, K)$ is internally stable.

It only remains to show that $K$ in (12) is indeed optimal. The first equality in (21) holds, since no closed-loop dynamics was used to derive it. To evaluate $x_{0}$ and $u_{2,0}$ consider the closed-loop dynamics (see Eq. (17) in Box I) at $k = 0$ and use the dynamics of the controller (12). Finally, evaluate the closed-loop $\mathcal{H}_2$-norm. We get

$\|T_{CL}\|_2^2 = \|\tilde{T}\|_2^2 + 2 \text{Trace}(\text{Re}(B^*X(\text{AE}^{-1}_F + J) \times B_F E^{-1}_E (B_1 + L^* D_2 B_1) + A_0 E^{-1}_F B_1)) \times B_1^* A B_1$, where $\tilde{T}_{CL} = \text{LFT}(T, K)$. The equality holds for all stabilizing controllers. Since $\tilde{T}$ satisfies the hypotheses of a two-block problem, $\min \|T_{CL}\|_2^2$ and therefore $\min \|T_{CL}\|_2^2$ are attained for $K$ in (12). This ends the whole proof.

References


