Port Controlled Hamiltonian systems and related nonlinear control approaches

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Sketch of the talk

1. Introduction to PHS
2. Port-Controlled Hamiltonian (PCH) systems
3. Implicit Port Hamiltonian systems
4. Physical approaches to the nonlinear control problem
5. Conclusions, prospects and opportunities
An introduction to distributed parameters systems
Complex water systems

Figure: Schematic view of the Bourne irrigation system near Valence
Tokamak fusion reactors

**Figure:** Schematic view of the coming ITER or existing Tore Supra tokamak reactors at CEA - Cadarache (Saint-Paul lez Durance)
High performance heating/refrigeration systems

Figure 1.1 System Diagram - Ideal Subcritical Vapor Compression Cycle
Figure 1.2 P-h Diagram - Ideal Subcritical Vapor Compression Cycle
Other examples of distributed parameters systems

- academic examples: wave, heat, beam, membrane and telegrapher equations
- piezoelectric actuators
- fluid flow and fluid/structure control systems
- magneto-hydrodynamic flows, plasma control
- classical Maxwell field equations
- quantum mechanics (Schrödinger and Klein-Gordon equations)
- free surface problems (Burger, Korteweg de Vries, Boussinesq, Saint-Venant/SWE)
- thermodynamics (chemical reactions, transport phenomena, phases equilibrium, etc.)
- biology (preys-predators, population dynamics, bio-reactors)

⇒ most not simplified real world applications lead to DPS models!
Some characteristics of distributed parameters systems

- Variables are **non uniform** in space, state equations are **Partial Differential Equations** (PDEs)
- **Boundary Control Systems** or **distributed control**
- Extensions of classical control results exist for linear DPS using **semigroup theory**: transfer functions, I/O operators, controllability or observability Grammians, state space realization, LQG or $H_\infty$ control, ...
- Results exist for the **regional analysis** and control of linear (or bilinear) DPS (regional controllability or observability, spreadability, viability, etc.)

- Practical solutions for control laws or observers design remain hard to achieve (solution of operators equations, infinite dimensional control)
- They are no general results for nonlinear DPS

⇒ **Particular cases**: systems of conservation laws with I/O or port variables
Different approaches for the control of DPS

- finite dimensional approximation (total or partial discretization)

- discrete modelling approaches (cellular automata, Lattice Boltzmann models)

- semigroup of linear operators (extensions to the semilinear case)
  [Curtain et Zwart 1995] *An introduction to infinite-dimensional linear systems theory*, Springer-Verlag

- PDE / functional analysis approach
  [Lions 1988] *Exact controllability, stabilizability and perturbations for distributed systems*, SIAM Rev. 30 pp. 1-68

- regional analysis (mainly using PDE/semigroup approaches)

- port-Hamiltonian approach
# The port-Hamiltonian approach: motivation

## Use physical insight explicitly
- modular modelling
- decomposition of simulation codes: reusability, parallel computation
- specific integration schemes conserving physical invariants
- physically-based control design (Lyapunov functions, passivity based approach, energy shaping)
- simultaneous design of process and control

## Inspired from thermodynamics and mechanics
- **Irreversible thermodynamics**: systems of balance equations and phenomenological laws
- **Analytical mechanics**: Hamiltonian variational formulations in \((q, p)\) coordinates
Finite dimensional Port-Controlled Hamiltonian (PCH) systems
Hamiltonian mechanics with port variables

### Hamilton equations

\[
\begin{pmatrix}
\frac{\partial q}{\partial t} \\
\frac{\partial p}{\partial t}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial H}{\partial p}(q, p) \\
\frac{\partial H}{\partial q}(q, p) + \tau
\end{pmatrix}
\]

with \( q \) the generalized coordinates, \( p \) the corresponding momenta, \( \tau \) the generalized forces, \( M(q) \) the inertia matrix, \( P(q) \) the potential energy and \( H(q, p) \) the total energy (Hamiltonian).

### Balance equation and passivity

\[
dH = \frac{\partial^T H}{\partial q} \dot{q} + \frac{\partial^T H}{\partial p} \dot{p} = \frac{\partial^T H}{\partial p} \tau = \dot{q}^T \tau = y^T(t) u(t)
\]

with \( y := \dot{q} \) and \( u := \tau \). Therefore if \( P(q) \) is bounded from below, the system is passive w.r.t. the pair of power-conjugated variables \((u, y)\) and the storage function \( H(q, p)\).
Conservative Port-Controlled Hamiltonian (PCH) systems

\[
\left( \begin{array}{c} \frac{\partial q}{\partial t} \\ \frac{\partial p}{\partial t} \end{array} \right) = \left( \begin{array}{c} \frac{\partial H}{\partial p}(q, p) \\ \frac{\partial H}{\partial q}(q, p) + \tau \end{array} \right)
\]

Generalization in local state-space coordinates

\[
\begin{align*}
\dot{x} &= J(x) \frac{\partial H}{\partial x} + g(x)u \\
y &= g^T(x) \frac{\partial H}{\partial x}
\end{align*}
\]

where \( u, y \in \mathbb{R}^m \) are power conjugated input-output variables and \( x = (x_1, \ldots, x_n) \) are local coordinates for the \( n \)-dimensional state space manifold \( \mathcal{X} \).
Poisson structure

The interconnection structure $J(x)$ usually satisfies

1. **skew-symmetry** $J(x) = -J^T(x)$ which implies conservativeness and **passivity** w.r.t. $(u, y)$ and the storage function $H$:

$$\frac{dH}{dt}(x(t)) = \frac{\partial^T H}{\partial x} \dot{x} = \frac{\partial^T H}{\partial x} \left( J(x) \frac{\partial H}{\partial x} + g(x)u \right) = y^T(t)u(t)$$

2. **Jacobi identities**

$$\sum_{l=1}^{n} \left[ J_{lj} \frac{\partial J_{ik}}{\partial x_l} + J_{li} \frac{\partial J_{kj}}{\partial x_l} + J_{lk} \frac{\partial J_{ji}}{\partial x_l} \right] = 0 \ \forall i, j, k$$

which guarantee **integrability** and existence of **canonical local coordinates** $\tilde{x} = (q, p, s)$ s.t. $J(x)$ reduces to (symplectic structure)

$$\tilde{J} = \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
PCH example: LCTG circuits

\[
\begin{bmatrix}
\dot{Q} \\
\dot{\varphi}_1 \\
\dot{\varphi}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & -1 \\
-1 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial H}{\partial \varphi_1} \\
\frac{\partial H}{\partial \varphi_2}
\end{bmatrix}
+ \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} u(t)
\]

where

\[H(\varphi_1, \varphi_2, Q) = \frac{\varphi_1^2}{2L_1} + \frac{\varphi_2^2}{2L_2} + \frac{Q^2}{2C}; \quad u(t) := V(t) ; \quad y(t) := \frac{\partial H}{\partial \varphi_1} = I\varphi_1\]
PCH systems with dissipation

Terminating some ports with resistive elements

\[ g(x)u = \begin{bmatrix} \tilde{g}(x) & g_R(x) \end{bmatrix} \begin{bmatrix} \tilde{u} \\ u_R \end{bmatrix} \]

\[ \begin{bmatrix} \tilde{y} \\ y_R \end{bmatrix} = \begin{bmatrix} \tilde{g}^T(x) \\ g_R^T(x) \end{bmatrix} \frac{\partial H}{\partial x}(x) \]

\[ u_R = -S(x)y_R \quad ; \quad S \geq 0 \]

leads to the input-state-output port-Hamiltonian formulation:

\[ \dot{x} = (J(x) - R(x)) \frac{\partial H}{\partial x} + \tilde{g}(x)\tilde{u} \]

\[ \tilde{y} = \tilde{g}^T(x) \frac{\partial H}{\partial x} \]

where \( R(x) := g_R(x)S(x)g_R^T(x) \geq 0 \) and

\[ \frac{dH}{dt}(x(t)) = y^T(t)u(t) - \frac{\partial^T H}{\partial x}R(x)\frac{\partial H}{\partial x} \leq y^T(t)u(t) \]
PCH example with dissipation: capacitor microphone

**Figure:** The capacitance $C(q)$ is varying with the displacement $q$ of the right plate with mass $m$. This plate is attached to a linear spring with constant $k$ and a linear damper with constant $c$. It is actuated by a mechanical force $F$ (air pressure arising from sound). $E$ is considered as a voltage source.
The dynamical equations of motion can be written as the port-Hamiltonian system with dissipation

\[
\begin{bmatrix}
\dot{q} \\
\dot{p} \\
\dot{Q}
\end{bmatrix}
= \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
q \\
p \\
Q
\end{pmatrix}
- \begin{pmatrix}
0 & 0 & 0 \\
0 & c & 0 \\
0 & 0 & 1/R
\end{pmatrix}
\begin{pmatrix}
\frac{\partial H}{\partial q} \\
\frac{\partial H}{\partial p} \\
\frac{\partial H}{\partial Q}
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 \\
1 & 0 \\
0 & 1/R
\end{pmatrix}
\begin{bmatrix}
\tilde{g} \\
F \\
E
\end{bmatrix}
\]

where

\[
H(\varphi_1, \varphi_2, Q) = \frac{\varphi_1^2}{2L_1} + \frac{\varphi_2^2}{2L_2} + \frac{Q^2}{2C} ; \quad u(t) := V(t) ; \quad y(t) := \frac{\partial H}{\partial \varphi_1} = I_{\varphi_1}
\]
Implicit Port Hamiltonian systems

- [Dorfman 1993] *Dirac Structures and Integrability of Nonlinear Evolution Equations*, Wiley
- [Cervera & al. 2007] *Interconnection of port-Hamiltonian systems and composition of Dirac structures*, Automatica, 43:212-225
The bond space of power conjugated effort and flow variables

Let $\mathcal{F}$ and $\mathcal{E}$ be two real vector spaces and assume that they are endowed with a non degenerated bilinear form, called **pairing** denoted by:

$$\langle \cdot | \cdot \rangle : \mathcal{F} \times \mathcal{E} \rightarrow \mathbb{R}$$

$$(f, e) \mapsto \langle e | f \rangle$$

(1)

On the product space, called **bond space**:

$$\mathcal{B} = \mathcal{F} \times \mathcal{E}$$

the bilinear product leads to the definition of a symmetric bilinear form, called **plus pairing** as follows:

$$\ll \cdot , \cdot \gg : \mathcal{B} \times \mathcal{B} \rightarrow \mathbb{R}$$

$$((f_1, e_1), (f_2, e_2)) \mapsto \ll (f_1, e_1), (f_2, e_2) \gg := \langle e_1 | f_2 \rangle + \langle e_2 | f_1 \rangle$$

(2)
Dirac structure on vector spaces

Definition

A Dirac structure is a linear subspace $\mathcal{D} \subset \mathcal{B}$ such that $\mathcal{D} = \mathcal{D}^\perp$, with $\perp$ denoting the orthogonal complement with respect to the bilinear form $\langle , \rangle$.

If a linear subspace $\mathcal{D} \subset \mathcal{B}$ satisfies only the isotropy condition $\mathcal{D} \subset \mathcal{D}^\perp$, what means that it is not maximal, one say that it is a Tellegen structure.

This name arises from the fact that the condition $\mathcal{D} \subset \mathcal{D}^\perp$ is equivalent to:

$$\langle e| f \rangle = 0, \ \forall (f, e) \in \mathcal{D}.$$ 

This condition is known as Tellegen’s theorem for the admissible voltages $e \in \mathbb{R}^n$ and currents $f \in \mathbb{R}^n$ of an electrical network, endowed with the Euclidean product.
Representations of finite-dimensional Dirac structures

Dirac structures admit algebraic definitions, based on some skew-symmetric linear maps called representation of a Dirac structure.


Assume that the flow and effort spaces are finite-dimensional and choose

\[ \mathcal{F} = \mathcal{E} = \mathbb{R}^n \]

with the power pairing being the canonical Euclidean product in \( \mathbb{R}^n \) composed with a signature matrix \( \sigma \):

\[ \langle e | f \rangle = e^T \sigma f \quad \text{where} \quad f \in \mathcal{F} = \mathbb{R}^n, \; e \in \mathcal{E} = \mathbb{R}^n \]  (3)
Theorem

A Dirac structure \( \mathcal{D} \subset \mathcal{B} = \mathbb{R}^n \times \mathbb{R}^n \) admits two \( n \times n \) real matrices, denoted here \( E \) and \( F \), and satisfying

\[
E \sigma F^T + F \sigma E^T = 0
\]

\[
\text{rank} [E : F] = n
\]

\( \mathcal{D} \) admits the image representation:

\[
\mathcal{D} = \{ (f, e) \in \mathcal{F} \times \mathcal{E} | f = E^T \lambda, \ e = F^T \lambda, \ \lambda \in \mathbb{R}^n \} \quad (4)
\]

and \( \mathcal{D} \) also admits the kernel representation:

\[
\mathcal{D} = \{ (f, e) \in \mathcal{F} \times \mathcal{E} | (F \sigma) f + (E \sigma) e = 0 \} \quad (5)
\]
Representations of a Dirac structure ... continued

**Theorem**

A Dirac structure $\mathcal{D} \subset \mathcal{B} = \mathbb{R}^n \times \mathbb{R}^n$ admits a decomposition of the flow and effort spaces:

$$
\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2 \ni \left( \begin{array}{c} f_1 \\ 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ f_2 \end{array} \right) \text{ and } \mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \ni \left( \begin{array}{c} e_1 \\ 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ e_2 \end{array} \right)
$$

and a skew-symmetric $n \times n$ real matrix denoted $J$ which define the input-output representation:

$$
\mathcal{D} = \left\{ \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) + \left( \begin{array}{c} e_1 \\ e_2 \end{array} \right) \right\} \in \mathcal{F} \times \mathcal{E} \mid \left( \begin{array}{c} f_1 \\ e_2 \end{array} \right) = J \left( \begin{array}{c} e_1 \\ f_2 \end{array} \right) \right\}
$$

(6)
Hamiltonian system defined w.r.t. a Dirac structure

We consider the case where the state space is a real vector space $\mathcal{F}$ of dimension $n$. Then the flow space is again $\mathcal{F}$ and the effort space will be chosen as its dual space $\mathcal{F}^*$. Then the bond space is simply $\mathcal{F} \times \mathcal{F}^*$. The power pairing is defined using the duality product:

$$P(t) = \langle f | e \rangle := \langle e, f \rangle \quad \forall (f, e) \in \mathcal{F} \times \mathcal{F}^*$$

**Definition**

Consider a Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$. A Hamiltonian system with respect to the Dirac structure $\mathcal{D}$ generated by the Hamiltonian function $H \in C^\infty (\mathcal{F}, \mathbb{R})$, is defined by the implicit differential equation:

$$\left( \frac{dx}{dt}, \frac{\partial H}{\partial x} \right) \in \mathcal{D}$$

The isotropy of the Dirac structure implies the conservation of the Hamiltonian:

$$\frac{dH}{dt} = \left\langle \frac{dx}{dt} \mid \frac{\partial H}{\partial x} \right\rangle = 0 \quad (7)$$
Example: closed LCTG circuits

Figure: A simple LC circuit composed of two capacitors and an inductor

Kirchhoff’s laws define the kernel representation of a Dirac structure:

\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\mathbf{i}_{C_1} \\
\mathbf{v}_L \\
\mathbf{i}_{C_2}
\end{pmatrix}
+ \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
1 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
\mathbf{v}_{C_1} \\
\mathbf{i}_L \\
\mathbf{v}_{C_2}
\end{pmatrix}
= 0
\] (8)
Example: LC circuit ... continued

The chosen state variables are:

\[ x = \begin{pmatrix} q_1 \\ \phi \\ q_2 \end{pmatrix} \]

charge of capacitor 1

flux in inductor

charge of capacitor 2

then \[ \frac{dx}{dt} = \begin{pmatrix} i_{C1} \\ v_L \\ v_{C2} \end{pmatrix} \in \mathcal{F} \]

The conjugated variables are the derivatives of the total electro-magnetic energy: \( H(x) \) w.r.t. the state variables, i.e.:

\[ \frac{\partial H}{\partial x}(x) = \begin{pmatrix} v_{C1} \\ i_L, \\ i_{C2} \end{pmatrix} \in \mathbb{R}^3 = \mathcal{E} \]
Example: LC circuit ... continued

The dynamics of the LC circuit is an implicit Hamiltonian system.

\[
\left( \frac{dx}{dt}, \frac{\partial H}{\partial x} \right) \in D \iff \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ \phi \\ q_2 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \frac{\partial H}{\partial x}(x) = 0
\]

(9)

The rank degeneracy of \( F \) corresponds to the constraint:

\[
\frac{\partial H}{\partial q_1}(x) - \frac{\partial H}{\partial q_2}(x) = 0
\]
Tangent and co-tangent spaces of the state space are augmented with a set of port variables used to represent the interaction of the system with its environment.

We augment the bond space according to

\[ \mathcal{B} := \left\{ \left( \begin{pmatrix} f^i \\ f^e \end{pmatrix}, \begin{pmatrix} e^i \\ e^e \end{pmatrix} \right) \in \mathcal{F} \times \mathcal{E} = \left( \mathcal{F}^i \times \mathcal{F}^e \right) \times \left( \mathcal{F}^i \times \mathcal{E}^e \right) \right\} \]

with the help of two other finite-dimensional port spaces \( \mathcal{F}^e \) and \( \mathcal{E}^e \) endowed with the non degenerated bilinear form \( \langle . | . \rangle_e \).

The augmented bond space \( \mathcal{B} \) is then endowed with the bilinear form:

\[ \langle \left( \begin{pmatrix} f^i \\ f^e \end{pmatrix}, \begin{pmatrix} e^i \\ e^e \end{pmatrix} \right) | \left( \begin{pmatrix} f^i \\ f^e \end{pmatrix}, \begin{pmatrix} e^i \\ e^e \end{pmatrix} \right) \rangle = \langle e^i | f^i \rangle_i + \langle e^e | f^e \rangle_e \]
Port Hamiltonian Systems

**Definition**

A **port Hamiltonian system** w.r.t. the Dirac structure $\mathcal{D} \subset \mathcal{B}$ and generated by the Hamiltonian function, $H \in C^\infty (\mathcal{F}^i, \mathbb{R})$ is defined by the implicit differential equation:

$$\left( \left( \begin{array}{c} \frac{dx}{dt} \\ f^e \end{array} \right), \left( \begin{array}{c} \frac{\partial H}{\partial x} \\ e^e \end{array} \right) \right) \in \mathcal{D}$$

- Port Hamiltonian systems **do not satisfy the Cauchy conditions** as long as the system is not completed with some relations on the external port variables $(f^e, e^e)$.

- Complex systems composed of a set of interacting subsystems may be described using the **Composition of Dirac structures** rather than interaction Hamiltonian functions.

- The isotropy property of the Dirac structure translates now into a **balance equation** for the Hamiltonian:

$$0 = \left\langle \frac{\partial H}{\partial x} \mid \frac{dx}{dt} \right\rangle + \left\langle e^e \mid f^e \right\rangle_e = \frac{dH}{dt} + \left\langle e^e \mid f^e \right\rangle_e \quad (10)$$
Example: open LCTG circuit

Kirchhoff’s laws define the kernel representation of an extended Dirac structure

\[
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
i_C \\
v_L \\
i_1 \\
i_2
\end{pmatrix}
+\begin{pmatrix}
0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 1 \\
1 & 0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
v_C \\
i_L \\
v_1 \\
v_2
\end{pmatrix} = 0
\]
The chosen state variables are:

\[ x = \begin{pmatrix} q \\ \phi \end{pmatrix} \]

charge of capacitor
flux in inductor

\[ \Rightarrow \frac{dx}{dt} = \begin{pmatrix} i_{C_1} \\ v_L \end{pmatrix} \in \mathcal{F}^i \]

The conjugated variables are the derivatives of the total electro-magnetic energy \( H(x) \) w.r.t. the state variables, i.e.:

\[ \frac{\partial H}{\partial x}(x) = \begin{pmatrix} v_C \\ i_L \end{pmatrix} \in \mathbb{R}^2 = \mathcal{F}^{i*} \]
Open LC circuit ... continued

The two open ports define the vector spaces of external currents $\mathcal{F}^e = \mathbb{R}^2 \ni (i_1, i_2)$ and voltages $\mathcal{E}^e = \mathbb{R}^2 \ni (v_1, v_2)$. Then the Dirac structure defined with the usual euclidian product is exactly

$\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{dq}{dt} \\
\frac{d\phi}{dt} \\
i_1 \\
i_2
\end{pmatrix}
+ \begin{pmatrix}
0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 1 \\
1 & 0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{\partial H}{\partial q} \\
\frac{\partial H}{\partial \phi} \\
v_1 \\
v_2
\end{pmatrix} = 0

(11)
Partial conclusion on implicit PCH systems

Port Hamiltonian systems are an extension of Hamiltonian systems which:

- are defined by extending the internal spaces of tangent and cotangent spaces by a pair of conjugated port vector spaces
- are using a Dirac structure on these extended effort and flow spaces
- are similar to (constrained) Poisson Hamiltonian systems
- allows to write balance equations including energy flows from the environment (hence to prove passivity properties)
- allows interconnection (including feedback control!) through composition of Dirac structures
Nonlinear control based on PCH systems properties

PCH system model (reminder)

\[
\sum : \begin{cases}
\dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x} (x) + g(x) u \\
y &= g^T(x) \frac{\partial H}{\partial x} (x)
\end{cases}
\]

where

- \( x \in \mathbb{R}^n \) are the energy state variables,
- \( H \) is the Hamiltonian function
- \( u, y \) are the input/output port variables
- \( J = -J^T \) a \( n \times n \) is the structure skew matrix matrix
- \( g \) is a \( n \times m \) input matrix
- \( R = R^T \geq 0 \) is the symmetric positive definite dissipation matrix.
Passivity and energy Balance

Energy balance

\[
\int_0^t u^T(\tau)y(\tau)d\tau = H[x(t)] - H[x(0)] + \int_0^t \left[ \frac{\partial H}{\partial x} \right]^T R \left[ \frac{\partial H}{\partial x} \right] d\tau
\]

- supplied
- stored
- dissipated energy

Passive system

\[
\frac{dH}{dt} \leq u^Ty
\]

- Let \( x_* = \arg \min H(x) \), set \( u = 0 \), then \( H(x) \) decreases and the system reaches \( x_* \). The system is locally stable.

- A Damping injection \( u = -Ky, (K = K^T > 0) \) increases the convergence rate to \( x_* \).
Given a desired equilibrium $x_d$, find a control in form $u = \beta(x) + v$ such that the closed loop dynamics satisfies:

$$H_d(x(t)) - H_d(x(0)) = \int_0^t v^T(\tau)z(\tau)d\tau - d(t)$$

- $H_d$ is the desired closed loop energy function
- $(v, z)$ is the new input-output passive pair of conjugated variables
- $d(t)$ is a dissipation term used to increase the convergence rate
First solution: the energy shaping method

**General principle**

If there exists $\beta(x)$ such that

$$- \int_0^t \beta^T(x(\tau)) y(\tau) \, d\tau = H_a(x) + \kappa$$

where $\kappa$ is a constant determined by the initial condition,

then the closed loop system with $u = \beta(x) + v$ has the total shaped energy

$$H_d(x) = H(x) + H_a(x)$$

If $x_d = \arg \min H_d(x)$, the system is stable at the desired equilibrium $x_d$.
Example (Series RLC circuit)

\[
\sum: \begin{cases} 
\dot{x}_1 &= \frac{1}{L} x_2 \\
\dot{x}_2 &= -\frac{1}{C} x_1 - \frac{R}{L} x_2 + u \\
y &= \frac{1}{L} x_2 
\end{cases}
\]

- If \( u = 0 \), the equilibrium is \( x_* = (0, 0) \)
- If \( u = V_d \), the equilibrium is \( x_d = (x_{1d}, 0), \ x_{1d} = CV_d \)

Shaped energy

Choosing

\[
H_d (x) = \frac{1}{2} \left( \frac{1}{C} + \frac{1}{C_a} \right) (x_1 - x_{1d})^2 + \frac{1}{2L} x_2^2 + \kappa
\]

with \( x_d = \text{argmin} \ (H_d (x)) \), one gets:

\[
H_a (x_1) = \frac{1}{2C_a} x_1^2 - \left( \frac{1}{C} + \frac{1}{C_a} \right) x_1 x_{1d}
\]

and the control law

\[
u = -\frac{x_1}{C_a} + \left( \frac{1}{C} + \frac{1}{C_a} \right) x_{1d}
\]
Drawbacks for the energy shaping method

- derivation of the control law from the shaped energy may be intricated and constrained
  → make use of the **PCH model structure** ("Control by interconnection")
- the condition for the existence of a power shaping control law is
  \[ R(x) \frac{\partial H_a}{\partial x}(x) = 0 \]
  → **dissipation obstacle**: \( H_a(x) \) should not depend on the coordinates where there is natural damping

The dissipation in energy balancing PBC is admissible only on the coordinates which does not require "shaping of energy"
Second solution: control by Interconnection (Cbi)

\[
\begin{align*}
\dot{\zeta} &= [J_C(\zeta) - R_C(\zeta)] \frac{\partial H_C}{\partial \zeta} (\zeta) + g_C(\zeta) u_C \\
y_C &= g_T(\zeta) \frac{\partial H_C}{\partial \zeta} (\zeta) \\
\begin{bmatrix}
u \\
u_C
\end{bmatrix} &= 
\begin{bmatrix}
0 & -l_m \\
l_m & 0
\end{bmatrix}
\begin{bmatrix}
y \\
y_C
\end{bmatrix} + 
\begin{bmatrix}
v \\
0
\end{bmatrix}
\end{align*}
\]

Figure: Diagram of the Cbi scheme

Closed loop system

\[
\begin{align*}
\dot{x} &= \begin{bmatrix}
J(x) - R(x) & -g(x) g_T(\zeta) \\
g_C(\zeta) g^T(x) & J_C(\zeta) - R_C(\zeta)
\end{bmatrix}
\begin{bmatrix}
\frac{\partial H_{cl}}{\partial x}(x, \zeta) \\
\frac{\partial H_{cl}}{\partial \zeta}(x, \zeta)
\end{bmatrix} \\
H_{cl}(x, \zeta) &= H(x) + H_C(\zeta) \quad \text{and} \quad \frac{\partial H_{cl}}{\partial t} \leq v^T y
\end{align*}
\]
Casimir function and static closed loop feedback

The closed loop Hamiltonian system may have Casimir’s functions $C(x, \zeta)$ such that

$$\frac{d}{dt} C(x(t), \zeta(t)) = 0$$

along the trajectories. It is then always possible to write Casimir’s functions in the form

$$C(x, \zeta) = \zeta - F(x) = 0$$

Dissipation obstacle ... again

Existence of a complete set of Casimir’s functions is related to integrability conditions. It requires

$$R(x) \frac{\partial H_c(F(x))}{\partial x} = 0$$
Parallel RLC circuit example

**Example**

A parallel RLC circuit

\[ V \]

\[ \begin{array}{c}
    L \\
    R \\
    C
\end{array} \]

\[ \sum : \begin{cases}
    \dot{x}_1 = -\frac{1}{RC}x_1 + \frac{1}{L}x_2 \\
    \dot{x}_2 = -\frac{1}{C}x_1 + u \\
    y = \frac{1}{L}x_2
\end{cases} \]

\[ H(x) = \frac{1}{2C}x_1^2 + \frac{1}{2L}x_2^2 \]

The dissipation structure is

\[ R(x) = \begin{pmatrix}
    \frac{1}{R} & 0 \\
    0 & 0
\end{pmatrix} \]

⇒ **Dissipation obstacle**

They are PBC methods which allow to overcome the dissipation obstacle:

- Cbl with a modified output to relax the DO condition [Ortega et al. 08]
- IDA-PBC control
The Interconnection and Damping Assignment Passivity Based Control

- uses a static feedback control law $u = \beta(x)$
- uses the PCH structure of the model

to get a closed loop Hamiltonian system in the form:

$$\dot{x} = [J_d(x) - R_d(x)] \frac{\partial H_d}{\partial x}(x)$$

with

- $J_d(x) = J(x) + J_a(x), \quad J_d = -J_d^T$, the desired interconnection
- $R_d(x) = R(x) + R_a(x), \quad R_d = R_d^T \geq 0$, the desired damping
- $H_d(x) = H(x) + H_a(x)$, the closed loop Hamiltonian with desired equilibrium properties
To achieve these desired structure, damping and Hamiltonian, we need a feedback law \( u = \beta(x) \) and a vector function \( K(x) = \frac{\partial H_a}{\partial x} \) which solve

- the **matching equation**:

\[
\left[ J_d (x) - R_d (x) \right] K(x) = - \left[ J_a (x) - R_a (x) \right] \frac{\partial H}{\partial x} (x) + g(x) \beta(x)
\]

- the **integrability equation**:

\[
\partial_x K(x) = \left[ \partial_x K(x) \right]^T
\]

- the **equilibrium assignment equation**:

\[
K(x_d) = -\partial_x H (x_d)
\]

- the **Lyapunov stability inequality**

\[
\partial_x K(x_d) > -\partial_{xx}^2 H (x_d)
\]
The closed loop system with $u = \beta(x)$ is (locally) stable at the desired equilibrium $x_d = \arg\min (H_d(x))$

$J_a(x)$ and $R_a(x)$ are free provided that the constraints $J_d = -J_d^T$ and $R_d = R_d^T \geq 0$ are satisfied.

**Controller design and control law**

When $g^\perp(x)$ s.t. $g^\perp(x) g(x) = 0$ exists, the matching equations reduces to

$$g^\perp(x) [J_d(x) - R_d(x)] K(x) = -g^\perp(x) [J_a(x) - R_a(x)] \frac{\partial H}{\partial x}(x)$$

and the control law may be explicitly derived

$$\beta(x) = \left[ g^T(x) g(x) \right]^{-1} g^T(x) \left\{ [J_d(x) - R_d(x)] K(x) + [J_a(x) - R_a(x)] \frac{\partial H}{\partial x}(x) \right\}$$
Parallel RLC circuit example

Model

\[ H(x) = \frac{1}{2C} x_1^2 + \frac{1}{2L} x_2^2 \]

\[
\sum: \begin{cases} 
\dot{x}_1 = -\frac{1}{RC} x_1 + \frac{1}{L} x_2 \\
\dot{x}_2 = -\frac{1}{C} x_1 + u \\
y = \frac{1}{L} x_2 
\end{cases}
\]

Control design

- natural damping \((R_a = 0)\) and interconnection structure \((J_a = 0)\)
- the matching equation reduces to
  \[ [J(x) - R(x)] \partial_x H_a = g(x) \beta(x) \]
- solutions are of the form
  \[ H_a(x) = \Phi(Rx_1 + x_2) \]
  where \(\Phi\) is any arbitrary integration function
Parallel RLC circuit example ... continued

Hamiltonian design

A choice

\[ H_a(x) = \frac{K_p}{2} \left[ (Rx_1 + x_2) - (Rx_{1d} + x_{2d}) \right]^2 - RV_d (Rx_1 + x_2) \]

with \( K_p > \frac{-1}{(L + CR^2)} \) leads to a closed loop Hamiltonian:

\[ H_d(x) = (x - x_d)^T \begin{bmatrix} \frac{1}{C} + R^2K_p & RK_p \\ RK_p & \frac{1}{L} + K_p \end{bmatrix} (x - x_d) + \kappa \]

with a global minimum \( x_d \). The control law is then

\[ u(x) = -K_p R \left[ R (x_1 - x_{1d}) + (x_2 - x_{2d}) \right] + R^2 V_d \]
Partial conclusion on PBC

- Passivity based control applies for all nonlinear physical (thermodynamical) systems.
- Dissipation obstacle may be overcome (Cbl, IDA-PBC).
- Structural approaches allow a deductive design of control laws and "Lyapunov" functions.
- Robustness to parameters uncertainties (only the qualitative energetic behaviour matters).
- Generalization are currently developed for distributed parameters systems.
- No parametrization of the solutions for the matching equation.
Extension and prospects in the field of nonlinear control for distributed parameters systems
An hyperbolic example: the shallow water equations (SWE)


State (energy) variables are chosen as differential forms:

\[ q(x, t) = \rho S(x, t) dx \quad \text{mass density} \]
\[ p(x, t) = \rho v(x, t) dx \quad \text{momentum density} \]

The energy density may be written in this SWE example

\[ H(x, t) = \rho \left( g \left( hA(x, h) - \int_0^h A(x, \xi) d\xi \right) + \frac{S(x, t) v^2(x, t)}{2} \right) dx \]
\[ = H(q, p) \]
The shallow water equations ... continued

**Conservation equations**

The simplified Saint-Venant equations is a system of 2 conservation laws

\[ \frac{\partial q}{\partial t} = - \frac{\partial}{\partial z} \left( S(x, t)v(x, t) \right) \, dx \quad \text{mass} \]

\[ \frac{\partial p}{\partial t} = - \frac{\partial}{\partial z} \left( \rho \left( gh(x, t) + \frac{v^2(x, t)}{2} \right) \right) \, dx \quad \text{momentum} \]

**Flow and effort variables**

*Flow* variables could be defined as the differential forms \((\dot{q}, \dot{p})\).

*Effort* variables are variational derivatives of \(H\) w.r.t. \(q(x, t)\) and \(p(x, t)\):

\[ e_p = \delta_p H = S(x, t)v(x, t) \quad \text{water flow} \]

\[ e_q = \delta_q H = \rho \left( gh(x, t) + \frac{v^2(x, t)}{2} \right) \quad \text{hydrodynamic pressure} \]
The shallow water equations ... continued

Hamiltonian formulation of SW conservation laws

The system of two conservation laws is defined using a very simple (canonical) 1D spatial interconnection structure with the help of these power conjugated flow \((f_q, f_p) := (\dot{q}, \dot{p})\) and effort \((e_q, e_p) = (\delta_q \mathcal{H}, \delta_p \mathcal{H})\) variables:

\[
- \frac{\partial}{\partial t} \begin{bmatrix} q(x, t) \\ p(x, t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} \delta_q \mathcal{H}(q, p) \\ \delta_p \mathcal{H}(q, p) \end{bmatrix}
\]
The shallow water equations ... continued

**Power balance equation**

\[
\frac{dH}{dt} = \frac{d}{dt} \int_Z \mathcal{H}(q, p) \\
= \int_Z \delta_q \mathcal{H} \wedge \dot{q} + \delta_p \mathcal{H} \wedge \dot{p} \\
= \int_Z e_q \wedge f_q + e_p \wedge f_p \\
= - \int_Z e_q \wedge d(e_p) + e_p \wedge d(e_q) \\
= - \int_Z d(e_q \wedge e_p) \\
= - \int_{\partial Z} e_{\partial} \wedge f_{\partial} \\
= p_d(0, t)Q(0, t) - p_d(L, t)Q(L, t)
\]

where \( e_{\partial} := e_q|_{\partial Z} \) and \( f_{\partial} := e_p|_{\partial Z} \)
The matrix differential operator

\[ \mathcal{J} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix} \]

is skew-symmetric. Indeed, consider two vectors of smooth functions \( e = (e_1, e_2) \) and \( e' = (e'_1, e'_2) \) satisfying the **homogeneous boundary conditions** \( e(a) = e(b) = e'(a) = e'(b) = 0 \), then:

\[
\int_a^b \left( e^t \mathcal{J} e' + e'^t \mathcal{J} e \right) dz = \int_a^b \left[ e_1 \left( \frac{\partial}{\partial z} e'_2 \right) + e_2 \left( \frac{\partial}{\partial z} e'_1 \right) + e_1 \left( \frac{\partial}{\partial z} e'_2 \right) + e_2 \left( \frac{\partial}{\partial z} e'_1 \right) \right] dz \\
= \left[ e_1 e'_2 + e_2 e'_1 \right]_a^b = 0
\]

- Hamiltonian density function
- Canonical skew-symmetric interconnection structure
- Power balance equation and boundary port variables

⇒ **PCH formalism for nonlinear distributed parameters systems!**
Ongoing research ...

### Port-Hamiltonian formulation
- moving boundary problems
- plasma dynamics in tokamak
- sediments and pollutants transport phenomena

### Discretization/reduction
- geometric reduction schemes
- I/O properties analysis using Grammians
- effective control computation

### Control
- spatial detection/localization problems (e.g. leaks in pipes)
- control of non linear parabolic systems
- control by interconnection of hyperbolic systems